Dynamic Equilibria in Time-Varying Networks*

Hoang Minh Pham and Leon Sering

Technische Universitt Berlin, Germany
hoang.m.pham@campus.tu-berlin.de sering@math.tu-berlin.de

Abstract. Predicting selfish behavior in public environments by considering Nash equilibria is a central concept of game theory. For the dynamic traffic assignment problem modeled by a flow over time game, in which every particle tries to reach its destination as fast as possible, the dynamic equilibria are called Nash flows over time. So far, this model has only been considered for networks in which each arc is equipped with a constant capacity, limiting the outflow rate, and with a transit time, determining the time it takes for a particle to traverse the arc. However, real-world traffic networks can be affected by temporal changes, for example, caused by construction works or special speed zones during some time period. To model these traffic scenarios appropriately, we extend the flow over time model by time-dependent capacities and time-dependent transit times. Our first main result is the characterization of the structure of Nash flows over time. Similar to the static-network model, the strategies of the particles in dynamic equilibria can be characterized by specific static flows, called thin flows with resetting. The second main result is the existence of Nash flows over time, which we show in a constructive manner by extending a flow over time step by step by these thin flows.

Keywords: Nash flows over time · dynamic equilibria · deterministic queuing · time-varying networks · dynamic traffic assignment.

1 Introduction

In the last decade the technological advances in the mobility and communication sector have grown rapidly enabling access to real-time traffic data and autonomous driving vehicles in the foreseeable future. One of the major advantages of self-driving and communicating vehicles is the ability to directly use information about the traffic network including the route-choice of other road users. This holistic view of the network can be used to decrease travel times and distribute the traffic volume more evenly over the network. As users will still expect to travel along a fastest route it is important to incorporate game theoretical aspects when analyzing the dynamic traffic assignment. The results can

* Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689).

then be used by network designers to identify bottlenecks beforehand, forecast air pollution in dense urban areas and give feedback on network structures. In order to obtain a better understanding of the complicated interplay between traffic users it is important to develop strong mathematical models which represent as many real-world traffic features as possible. Even though the more realistic models consider a time-component, the network properties are considered to stay constant in most cases. Surely, this is a serious drawback as real road networks often have properties that vary over time. For example, the speed limit in school zones is often reduced during school hours, roads might be completely or partially blocked due to construction work and the direction of reversible lanes can be switched, causing a change in the capacity in both directions. A more exotic, but nonetheless important setting are evacuation scenarios. Consider an inhabited region of low altitude with a high risk of flooding. As soon as there is a flood warning everyone needs to be evacuated to some high-altitude-shelter. But, due to the nature of rising water levels, roads with low altitude will be impassable much sooner than roads of higher altitude. In order to plan an optimal evacuation or simulate a chaotic equilibrium scenario it is essential to use a model with time-varying properties. This research work is dedicated to providing a better understanding of the impact of dynamic road properties on the traffic dynamics in the Nash flow over time model. We will transfer all essential properties of Nash flows over time in static networks to networks with time-varying properties.

1.1 Related Work

The fundamental concept for the model considered in this paper are *flows over time* or *dynamic flows*, which were introduced back in 1956 by Ford and Fulkerson [8,9] in the context of optimization problems. The key idea is to add a time-component to classical network flows, which means that the flow particles need time to travel through the network. In 1959 Gale [10] showed the existence of so called *earliest arrival flows*, which solve several optimization problems at once, as they maximize the amount of flow reaching the sink at all points in time simultaneously. Further work on these optimal flows is due to Wilkinson [26], Fleischer and Tardos [7], Minieka [18] and many others. For formal definitions and a good overview of optimization problems in flow over time settings we refer to the survey of Skutella [23].

In order to use flows over time for traffic modeling it is important to consider game theoretic aspects. Some pioneer work goes back to Vickrey [24] and Yagar [27]. In the context of classical (static) network flows, equilibria were introduced by Wardrop [25] in 1952. In 2009 Koch and Skutella [15] (see also [16] and Koch's PhD thesis [14]) started a fruitful research line by introducing dynamic equilibria, also called *Nash flows over time*, which will be the central concept in this paper. In a Nash flow over time every particle chooses a quickest path from the origin to the destination, anticipating the route choice of all other flow particles. Cominetti et al. showed the existence of Nash flows over time [3,4] and studied the long term behavior [5]. Macko et al. [17] studied the Braess paradox in this model and Bhaskar et al. [1] and Correa et al. [6] bounded the price of anarchy under certain conditions. In 2018 Sering and Skutella [21] transferred Nash flows over time to a model with multiple sources and multiple sinks and in the following year Sering and Vargas Koch [22] considered Nash flows over time in a model with spillback.

A different equilibrium concept in the same model was considered by Graf et al. [11] by introducing instantaneous dynamic equilibria. In these flows over time the particles do not anticipate the further evolution of the flow, but instead reevaluate their route choice at every node and continue their travel on a current quickest path. In addition to that, there is an active research line on packet routing games. Here, the traffic agents are modeled by atomic packets (vehicles) of a specific size. This is often combined with discrete time steps. Some of the recent work on this topic is due to Cao et al. [2], Harks et al. [12], Peis et al. [19] and Scarsini et al. [20].

1.2 Overview and Contribution

In the *base model*, which was considered by Koch and Skutella [16] and by the follow up research [1,3,4,5,6,17,21], the network is constant and each arc has a constant capacity and constant transit time. In real-world traffic, however, temporary changes of the infrastructure are omnipresent. In order to represent this, we extend the base model to networks with time-varying capacities (including the network inflow rate) and time-varying transit times.

We start in Section 2 by defining the flow dynamics of the deterministic queuing model with time-varying arc properties and proving some first auxiliary results. In particular, we describe how to turn time-dependent speed limits into time-dependent transit times. In Section 3 we introduce some essential properties, such as the earliest arrival times, which enable us to define Nash flows over time. As in the base model, it is still possible to characterize such a dynamic equilibrium by the underlying static flow. Taking the derivatives of these parametrized static flows provides thin flows with resetting, which are defined in Section 4. We show that the central results of the base model transfer to time-varying networks, and in particular, that the derivatives of every Nash flow over time form a thin flow with resetting. In Section 5 we show the reverse of this statement: Nash flows over time can be constructed by a sequence of thin flows with resetting, which, in the end, proves the existence of dynamic equilibria. We close this section with a detailed example. Finally, in Section 6 we present a conclusion and give a brief outlook on further research directions.

2 Flow Dynamics

We consider a directed graph $G = (V, E)$ with a source s and a sink t such that each node is reachable by s. In contrast to the Koch-Skutella model, which we will call *base model* from now on, this time each arc e is equipped with a time-dependent capacity $\nu_e \colon [0, \infty) \to (0, \infty)$ and a time-dependent speed limit $\lambda_e \colon [0, \infty) \to (0, \infty)$, which is inversely proportional to the transit time.

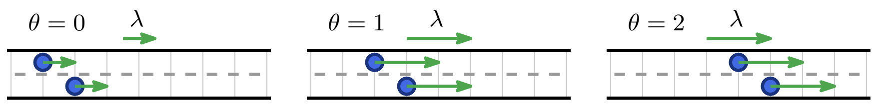

Fig. 1. Consider a road segment with time-dependent speed limit that is low in the time interval $[0, 1)$ and large afterwards. All vehicles, independent of their position, first traverse the link slowly and immediately speed up to the new speed limit at time 1.

We consider a time-dependent network inflow rate $r\colon [0,\infty) \to [0,\infty)$ denoting the flow rate at which particles enter the network at s. We assume that the amount of flow an arc can support is unbounded and that the network inflow is unbounded as well, i.e., for all $e \in E$ we require that

$$\int_0^\theta \nu_e(\xi)\,\mathrm{d}\xi \to \infty, \quad \int_0^\theta \lambda_e(\xi)\,\mathrm{d}\xi \to \infty \quad \text{and} \quad \int_0^\theta r(\xi)\,\mathrm{d}\xi \to \infty \quad \text{for } \theta \to \infty.$$

Later on, in order to be able to construct Nash flows over time, we will additionally assume that all these functions are right-constant, i.e., for every $\theta \in [0, \infty)$ there exists an $\varepsilon > 0$ such that the function is constant on $[\theta, \theta + \varepsilon)$.

Speed limits. Let us focus on the transit times first. We have to be careful how to model the transit time changes, since we do not want to lose the following two properties of the base model:

(i) We want to have the first-in-first-out (FIFO) property for arcs, which leads to FIFO property of the network for Nash flows over time [16, Theorem 1].
(ii) Particles should never have the incentive to wait on a node.

In other words, we cannot simply allow piecewise-constant transit times, since this could lead to the following case: If the transit time of an arc is high at the beginning and gets reduced to a lower value at some later point in time, then particles might overtake other particles on that arc. Thus, particles might arrive earlier at the sink if they wait right in front of the arc until its transit time drops. Hence, we let the speed limit change over time instead. In order to keep the number of parameters of the network as small as possible, we assume that the lengths of all arcs equal 1 and, instead of a transit time, we equip every arc $e \in E$ with a time-dependent speed limit $\lambda_e\colon [0,\infty) \to (0,\infty)$. Thus, a particle might traverse the first part of an arc at a different speed than the remaining distance if the maximal speed changes midway; see Figure 1.

Transit times. Note that we assume the point queue of an arc to always right in front of the exit. Hence, a particle entering arc e at time θ immediately traverses the arc of length 1 with a time-dependent speed of λ_e. The *transit time* $\tau\colon [0,\infty) \to [0,\infty)$ is therefore given by

$$\tau_e(\theta) \coloneqq \min\left\{ \tau \geq 0 \;\middle|\; \int_\theta^{\theta+\tau} \lambda_e(\xi)\,\mathrm{d}\xi = 1 \right\}.$$

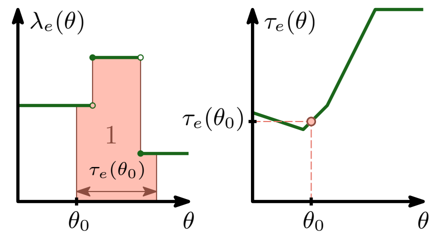

Fig. 2. From speed limits (*left side*) to transit times (*right side*). The transit time $\tau_e(\theta)$ denotes the time a particle needs to traverse the arc when entering at time θ. We normalize the speed limits by assuming that all arcs have length 1, and hence, the transit time $\tau_e(\theta)$ equals the length of an interval starting at θ such that the area under the speed limit graph within this interval is 1.

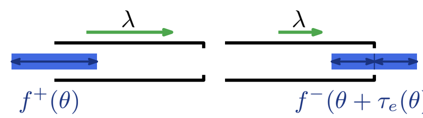

Fig. 3. An illustration of how the flow rate changes depending on the speed limits. *On the left:* As the speed limit λ is high, the flow volume entering the arc per time unit is represented by the area of the long rectangle. *On the right:* The speed limit is halved, and therefore, the same amount of flow needs twice as much time to leave the arc (or enter the queue if there is one). Hence, if there is no queue, the outflow rate at time $\tau + \tau_e(\theta)$ is only half the size of the inflow rate at time θ.

Since we required $\int_0^\theta \lambda_e(\xi)\,\mathrm{d}\xi$ to be unbounded for $\theta \to \infty$, we always have a finite transit time. For an illustrative example see Figure 2.

The following lemma shows some basic properties of the transit times.

Lemma 1. *For all $e \in E$ and almost all $\theta \in [0, \infty)$ we have:*

(i) The function $\theta \mapsto \theta + \tau_e(\theta)$ is strictly increasing.
(ii) The function τ_e is continuous and almost everywhere differentiable.
(iii) For almost all $\theta \in [0, \infty)$ we have $1 + \tau_e'(\theta) = \frac{\lambda_e(\theta)}{\lambda_e(\theta + \tau_e(\theta))}$.

These statement follow by simple computation and some basic Lebesgue integral theorems. The proof can be found in the appendix on page 16.

Speed ratios. The ratio in Lemma 1 (iii) will be important to measure the outflow of an arc depending on the inflow. We call $\gamma_e \colon [0, \infty) \to [0, \infty)$ the *speed ratio* of e and it is defined by $\gamma_e(\theta) := \frac{\lambda_e(\theta)}{\lambda_e(\theta + \tau_e(\theta))} = 1 + \tau_e'(\theta)$. Figuratively speaking, this ratio describes how much the flow rate changes under different speed limits. If, for example, $\gamma_e(\theta) = 2$, as depicted in Figure 3, this means that the speed limit was twice as high when the particle entered the arc as it is at the moment the particle enters the queue. In this case the flow rate is halved on its way, since the same amount of flow that entered within one time unit, needs two time units to leave it. With the same intuition the flow rate is increased whenever $\gamma_e(\theta) < 1$. Note that in figures of other publications on flows over time the flow rate is often pictured by the width of the flow. But for time-varying networks this is not accurate anymore as the transit speed can vary. Hence, in this paper the flow rates are given by the width of the flow multiplied by the current speed limit.

A *flow over time* is specified by a family of locally integrable and bounded functions $f = (f_e^+, f_e^-)_{e \in E}$ denoting the in- and outflow rates. The *cumulative in- and outflows* are given by

$$F_e^+(\theta) \coloneqq \int_0^\theta f_e^+(\xi)\,\mathrm{d}\xi \qquad \text{and} \qquad F_e^-(\theta) \coloneqq \int_0^\theta f_e^-(\xi)\,\mathrm{d}\xi.$$

A flow over time *conserves flow on all arcs* e:

$$F_e^-(\theta + \tau_e(\theta)) \leq F_e^+(\theta) \qquad \text{for all } \theta \in [0, \infty], \tag{1}$$

and *conserves flow at every node* $v \in V \setminus \{ t \}$ for almost all $\theta \in [0, \infty)$:

$$\sum_{e \in \delta_v^+} f_e^+(\theta) - \sum_{e \in \delta_v^-} f_e^-(\theta) = \begin{cases} 0 & \text{if } v \in V \setminus \{ t \}, \\ r(\theta) & \text{if } v = s. \end{cases} \tag{2}$$

A particle entering an arc e at time θ reaches the head of the arc at time $\theta + \tau_e(\theta)$ where it lines up at the point queue. Thereby, the *queue size* $z_e \colon [0, \infty) \to [0, \infty)$ at time $\theta + \tau_e(\theta)$ is defined by $z_e(\theta + \tau_e(\theta)) \coloneqq F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta))$.

We call a flow over time in a time-varying network *feasible* if we have for almost all $\theta \in [0, \infty)$ that

$$f_e^-(\theta + \tau_e(\theta)) = \begin{cases} \nu_e(\theta + \tau_e(\theta)) & \text{if } z_e(\theta + \tau_e(\theta)) > 0, \\ \min\left\{ \frac{f_e^+(\theta)}{\gamma_e(\theta)}, \nu_e(\theta + \tau_e(\theta)) \right\} & \text{else,} \end{cases} \tag{3}$$

and $f_e^-(\theta) = 0$ for almost all $\theta < \tau_e(0)$.

Note that the outflow rate depends on the speed ratio $\gamma_e(\theta)$ if the queue is empty (see Figure 3). Otherwise, the particles enter the queue, and therefore, the outflow rate equals the capacity independent of the speed ratio. Furthermore, we observe that every arc with a positive queue always has a positive outflow, since the capacities are required to be strictly positive. And finally, (3) implies (1), which can easily be seen by considering the derivatives of the cumulative flows whenever we have an empty queue, i.e., $F_e^-(\theta + \tau_e(\theta)) = F_e^+(\theta)$. By (3) we have that $f_e^-(\theta + \tau_e(\theta)) \cdot (1 + \tau_e'(\theta)) \leq f_e^+(\theta)$. Hence, (2) and (3) are sufficient for a family of functions $f = (f_e^+, f_e^-)_{e \in E}$ to be a feasible flow over time.

The *waiting time* $q_e \colon [0, \infty) \to [0, \infty)$ of a particle that enters the arc at time θ is defined by

$$q_e(\theta) \coloneqq \min \left\{ q \geq 0 \;\middle|\; \int_{\theta + \tau_e(\theta)}^{\theta + \tau_e(\theta) + q} \nu_e(\xi)\,\mathrm{d}\xi = z_e(\theta + \tau_e(\theta)) \right\}.$$

As we required $\int_0^\theta \nu_e(\xi)\,\mathrm{d}\xi$ to be unbounded for $\theta \to \infty$ the set on the right side is never empty. Hence, $q_e(\theta)$ is well-defined and has a finite value. In addition, q_e is continuous since ν_e is always strictly positive. The *exit time* $T_e \colon [0, \infty) \to [0, \infty)$ denotes the time at which the particles that have entered the arc at time θ finally leave the queue. Hence, we define $T_e(\theta) \coloneqq \theta + \tau_e(\theta) + q_e(\theta)$. In Figure 4 we display an illustrative example for the definition of waiting and exit times.

With these definitions we can show the following lemma.

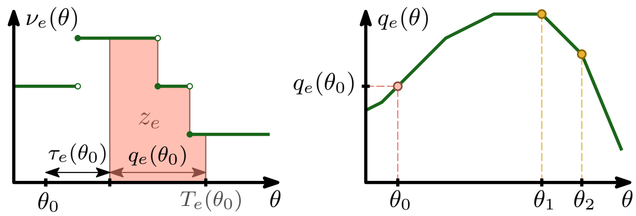

Fig. 4. Waiting times for time-dependent capacities. The waiting time of a particle θ_0 (*right side*) is given by the length of the interval starting at $\theta_0 + \tau_e(\theta_0)$ such that the area underneath the capacity graph equals the queue size at time $\theta_0 + \tau_e(\theta_0)$ (*left side*). The right boundary of the interval equals the exit time $T_e(\theta_0)$. The waiting time does not only depend on the capacity but also on the inflow rate and the transit times. For example, if the capacity and the speed limit are constant but the inflow rate is 0, the waiting time will decrease with a slope of 1 (*right side* within $[\theta_1, \theta_2]$).

Lemma 2. *For a feasible flow over time f it holds for all $e \in E$, $v \in V$ and $\theta \in [0, \infty)$ that:*

(i) $q_e(\theta) > 0 \quad \Leftrightarrow \quad z_e(\theta + \tau_e(\theta)) > 0$.
(ii) $z_e(\theta + \tau_e(\theta) + \xi) > 0$ *for all* $\xi \in [0, q_e(\theta))$.
(iii) $F_e^+(\theta) = F_e^-(T_e(\theta))$.
(iv) For $\theta_1 < \theta_2$ *with* $F_e^+(\theta_2) - F_e^+(\theta_1) = 0$ *and* $z_e(\theta_2 + \tau_e(\theta_2)) > 0$ *we have* $T_e(\theta_1) = T_e(\theta_2)$.
(v) The functions T_e are monotonically increasing.
(vi) The functions q_e and T_e are continuous and almost everywhere differentiable.
(vii) For almost all $\theta \in [0, \infty)$ we have

$$T_e'(\theta) = \begin{cases} \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} & \text{if } q_e(\theta) > 0, \\ \max\left\{ \gamma_e(\theta), \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} \right\} & \text{else.} \end{cases}$$

Most of the statements follow immediately from the definitions and some involve minor calculations. For (vi) we use Lebesgue's differentiation theorem. As the proof does not give any interesting further insights we moved it to the appendix on page 17.

3 Nash Flows Over Time

In order to define a dynamic equilibrium we consider the particles as players in a dynamic game. For this the set of particles is identified by the non-negative reals denoted by $\mathbb{R}_{\geq 0}$. The flow volume is hereby given by the Lebesgue-measure, which means that $[a, b] \subseteq \mathbb{R}_{\geq 0}$ with $a < b$ contains a flow volume of $b - a$. The flow particles enter the network according to the ordering of the reals beginning with particle 0. It is worth noting that a particle $\phi \in \mathbb{R}_{\geq 0}$ can be split up further so that for example one half takes a different route than the other half. As characterized by Koch and Skutella, a dynamic equilibrium is a feasible flow over time, where almost all particles only use current shortest paths from s to t. Note that we assume a game with full information. Consequently, all particles know all speed limit and capacity functions in advance and have the ability to perfectly predict the future evolution of the flow over time. Hence, each particle perfectly knows all travel times and can choose its route accordingly. We start by defining the earliest arrival times for a particle $\phi \in \mathbb{R}_{\geq 0}$.

The *earliest arrival time functions* $\ell_v \colon \mathbb{R}_{\geq 0} \to [0, \infty)$ map each particle ϕ to the earliest time $\ell_v(\phi)$ it can possibly reach node v. Hence, it is the solution to

$$\ell_v(\phi) = \begin{cases} \min \left\{ \theta \geq 0 \;\middle|\; \int_0^\theta r(\xi)\,\mathrm{d}\xi = \phi \right\} & \text{for } v = s, \\ \min\limits_{e = uv \in \delta_v^-} T_e(\ell_u(\phi)) & \text{else.} \end{cases} \tag{4}$$

Note that for all $v \in V$ the earliest arrival time function ℓ_v is non-decreasing, continuous and almost everywhere differentiable. This holds directly for ℓ_s and for $v \neq s$ it follows inductively, since these properties are preserved by the concatenation $T_e \circ \ell_u$ and by the minimum of finitely many functions.

For a particle ϕ we call an arc $e = uv$ *active* if $\ell_v(\phi) = T_e(\ell_u(\phi))$. The set of all these arcs are denoted by E'_ϕ and these are exactly the arcs that form the current shortest paths from s to some node v. For this reason we call the subgraph $G'_\phi = (V, E'_\phi)$ the *current shortest paths network* for particle ϕ. Note that G'_ϕ is acyclic and that every node is reachable by s within this graph. The arcs where particle ϕ experiences a waiting time when traveling along shortest paths only are called *resetting arcs* denoted by $E^*_\phi \coloneqq \{\, e = uv \in E \mid q_e(\ell_u(\phi)) > 0 \,\}$.

Nash flows over time in time-varying networks are defined in the exact same way as Cominetti et al. defined them in the base model [3, Definition 1].

Definition 1 (Nash flow over time). *We call a feasible flow over time f a* Nash flow over time *if the following* Nash flow condition *holds:*

$$f_e^+(\theta) > 0 \;\;\Rightarrow\;\; \theta \in \ell_u(\Phi_e) \quad \textit{for all } e = uv \in E \textit{ and almost all } \theta \in [0, \infty), \tag{N}$$

where $\Phi_e \coloneqq \{\, \phi \in \mathbb{R}_{\geq 0} \mid e \in E'_\phi \,\}$ is the set of particles for which arc e is active.

As Cominetti et al. showed in [4, Theorem 1] these Nash flows over time can be characterized as follows.

Lemma 3. *A feasible flow over time f is a Nash flow over time if, and only if, for all $e = uv \in E$ and all $\phi \in \mathbb{R}_{\geq 0}$ we have $F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi))$.*

Since the exit and the earliest arrival times have the same properties in time-varying networks as in the base model, this lemma follows with the exact same proof that was given by Cominetti et al. for the base model [4, Theorem 1]. The same is true for the following lemma; see [4, Proposition 2].

Lemma 4. *Given a Nash flow over time the following holds for all particles ϕ:*

(i) $E^*_\phi \subseteq E'_\phi$.
(ii) $E'_\phi = \{\, e = uv \mid \ell_v(\phi) \geq \ell_u(\phi) + \tau_e(\theta) \,\}$.
(iii) $E^*_\phi = \{\, e = uv \mid \ell_v(\phi) > \ell_u(\phi) + \tau_e(\theta) \,\}$.

Motivated by Lemma 3 we define the *underlying static flow* for $\phi \in \mathbb{R}_{\geq 0}$ by

$$x_e(\phi) \coloneqq F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi)) \quad \text{for all } e = uv \in E.$$

By the definition of ℓ_s and the integration of (2) we have $\int_0^{\ell_s(\phi)} r(\xi)\,\mathrm{d}\xi = \phi$, and hence, $x_e(\phi)$ is a static s-t-flow (classical network flow) of value ϕ, whereas the derivatives $(x'_e(\phi))_{e \in E}$ form a static s-t-flow of value 1.

4 Thin Flows

Thin flows with resetting, introduced by Koch and Skutella [16], characterize the derivatives $(x'_e)_{e \in E}$ and $(\ell'_v)_{v \in V}$ of Nash flows over time in the base model. In the following we will transfer this concept to time-varying networks.

Consider an acyclic network $G' = (V, E')$ with a source s and a sink t, such that every node is reachable by s. Each arc is equipped with a capacity $\nu_e > 0$ and a speed ratio $\gamma_e > 0$. Furthermore, we have a network inflow rate of $r > 0$ and an arc set $E^* \subseteq E'$. We obtain the following definition.

Definition 2 (Thin flow with resetting in a time-varying network). *A static s-t flow $(x'_e)_{e \in E}$ of value 1 together with a node labeling $(\ell'_v)_{v \in V}$ is a* thin flow with resetting *on E^* if:*

$$\ell'_s = \frac{1}{r} \tag{TF1}$$

$$\ell'_v = \min_{e = uv \in E'} \rho_e(\ell'_u, x'_e) \quad \textit{for all } v \in V \setminus \{\, s \,\}, \tag{TF2}$$

$$\ell'_v = \rho_e(\ell'_u, x'_e) \quad \textit{for all } e = uv \in E' \textit{ with } x'_e > 0, \tag{TF3}$$

$$\textit{where} \qquad \rho_e(\ell'_u, x'_e) \coloneqq \begin{cases} \frac{x'_e}{\nu_e} & \textit{if } e = uv \in E^*, \\ \max\left\{\, \gamma_e \cdot \ell'_u, \frac{x'_e}{\nu_e} \,\right\} & \textit{if } e = uv \in E' \backslash E^*. \end{cases}$$

The derivatives of a Nash flow over time in time-varying networks do indeed form a thin flow with resetting as the following theorem shows.

Theorem 1. *For almost all $\phi \in \mathbb{R}_{\geq 0}$ the derivatives $(x'_e(\phi))_{e \in E'_\phi}$ and $(\ell'_v(\phi))_{v \in V}$ of a Nash flow over time $f = (f_e^+, f_e^-)_{e \in E}$ form a thin flow with resetting on E^*_ϕ in the current shortest paths network $G'_\phi = (V, E'_\phi)$ with network inflow rate $r(\ell_s(\phi))$ as well as capacities $\nu_e(\ell_v(\phi))$ and speed ratios $\gamma_e(\ell_u(\phi))$ for each arc $e = uv \in E$.*

Proof. Let $\phi \in \mathbb{R}_{\geq 0}$ be a particle such that for all arcs $e = uv \in E$ the derivatives of x_e, ℓ_u, $T_e \circ \ell_u$ and τ_e exist and $x'_e(\phi) = f^+_e(\ell_u(\phi)) \cdot \ell'_u(\phi) = f^-_e(\ell_v(\phi)) \cdot \ell'_v(\phi)$ as well as $1 + \tau'_e(\ell_u(\phi)) = \gamma_e(\ell_u(\phi))$. This is given for almost all ϕ.

By (4) we have $\int_0^{\ell_s(\phi)} r(\xi)\,\mathrm{d}\xi = \phi$ and taking the derivative by applying the chain rule, yields $r(\ell_s(\phi)) \cdot \ell'_s(\phi) = 1$, which shows (TF1).

Taking the derivative of (4) at time $\ell_u(\phi)$ by using the differentiation rule for a minimum (Lemma 7 in the appendix) yields $\ell'_v(\phi) = \min_{e=uv\in E'} T'_e(\ell_u(\phi)) \cdot \ell'_u(\phi)$. By using Lemma 2 (vii) we obtain

$$\begin{aligned} T'_e(\ell_u(\phi)) \cdot \ell'_u(\phi) &= \begin{cases} \frac{f^+_e(\ell_u(\phi))}{\nu_e(T_e(\ell_u(\phi)))} \cdot \ell'_u(\phi) & \text{if } q_e(\ell_u(\phi)) > 0, \\ \max\left\{ \gamma_e(\ell_u(\phi)), \frac{f^+_e(\ell_u(\phi))}{\nu_e(T_e(\ell_u(\phi)))} \right\} \cdot \ell'_u(\phi) & \text{else}, \end{cases} \\ &= \rho_e(\ell'_u(\phi), x'_e(\phi)), \end{aligned}$$

which shows (TF2).

Finally, in the case of $f^-_e(\ell_v(\phi)) \cdot \ell'_v(\phi) = x'_e(\phi) > 0$ we have by (3) that

$$\begin{aligned} \ell'_v(\phi) = \frac{x'_e(\phi)}{f^-_e(\ell_v(\phi))} &= \begin{cases} \dfrac{x'_e(\phi)}{\min\left\{ \frac{f^+_e(\ell_u(\phi))}{\gamma_e(\ell_u(\phi))}, \nu_e(\ell_v(\phi)) \right\}} & \text{if } q_e(\ell_u(\phi)) = 0, \\ \dfrac{x'_e(\phi)}{\nu_e(\ell_v(\phi))} & \text{else}, \end{cases} \\ &= \begin{cases} \max\left\{ \gamma_e(\ell_u(\phi)) \cdot \ell'_u(\phi), \frac{x'_e(\phi)}{\nu_e(\ell_v(\phi))} \right\} & \text{if } e \in E'_\phi \backslash E^*_\phi, \\ \frac{x'_e(\phi)}{\nu_e(\ell_v(\phi))} & \text{if } e \in E^*_\phi, \end{cases} \\ &= \rho_e(\ell'_u(\phi), x'_e(\phi)). \end{aligned}$$

This shows (TF3) and finishes the proof. □

In order to construct Nash flows over time in time-varying networks, we first have to show that there always exists a thin flow with resetting.

Theorem 2. *Consider an acyclic graph $G' = (V, E')$ with source s, sink t, capacities $\nu_e > 0$, speed ratios $\gamma_e > 0$ and a subset of arcs $E^* \subseteq E'$, as well as a network inflow $r > 0$. Furthermore, suppose that every node is reachable from s. Then there exists a thin flow $((x'_e)_{e\in E}, (\ell'_v)_{v\in V})$ with resetting on E^*.*

This proof works exactly as the proof for the existence of thin flows in the base model presented by Cominetti et al. [4, Theorem 3]. In addition, a detailed proof utilizing Kakutani's fixed point theorem is given in the appendix.

5 Constructing Nash Flows Over Time

In the remaining part of this paper we assume that for all $e \in E$ the functions ν_e and λ_e as well as the network inflow rate function r are right-constant. In order to show the existence of Nash flows over time in time-varying networks we use the same α-extension approach as used by Koch and Skutella in [16] for the base model. The key idea is to start with the empty flow over time and expand it step by step by using a thin flow with resetting.

Given a *restricted Nash flow over time* f on $[0, \phi]$, i.e., a Nash flow over time where only the particles in $[0, \phi]$ are considered, we obtain well-defined earliest arrival times $(\ell_v(\phi))_{v \in V}$ for particle ϕ. Hence, by Lemma 4 we can determine the current shortest paths network $G'_\phi = (V, E'_\phi)$ with the resetting arcs E^*_ϕ, the capacities $\nu_e(\ell_v(\phi))$ and speed ratios $\gamma_e(\ell_u(\phi))$ for all arcs $e = uv \in E'$ as well as the network inflow rate $r(\ell_s(\phi))$. By Theorem 2 there exists a thin flow $((x'_e)_{e \in E'}, (\ell'_v)_{v \in V})$ on G'_ϕ with resetting on E^*_ϕ. For $e \notin E'_\phi$ we set $x'_e := 0$. We extend the ℓ- and x-functions for some $\alpha > 0$ by

$$\ell_v(\phi + \xi) := \ell_v(\phi) + \xi \cdot \ell'_v \quad \text{and} \quad x_e(\phi) := x_e(\phi) + \xi \cdot x'_e \qquad \text{for all } \xi \in [0, \alpha)$$

and the in- and outflow rate functions by

$$f^+_e(\theta) := \frac{x'_e}{\ell'_u} \text{ for } \theta \in [\ell_u(\phi), \ell_u(\phi{+}\alpha)); \qquad f^-_e(\theta) := \frac{x'_e}{\ell'_v} \text{ for } \theta \in [\ell_v(\phi), \ell_v(\phi{+}\alpha)).$$

We call this extended flow over time *α-extension*. Note that $\ell'_u = 0$ means that $[\ell_u(\phi), \ell_u(\phi + \alpha))$ is empty, and the same holds for ℓ'_v.

An α-extension is a restricted Nash flow over time, which we will prove later on, as long as the α stays within reasonable bounds. Similar to the base model we have to ensure that resetting arcs stay resetting and non-active arcs stay non-active for all particles in $[\phi, \phi + \alpha)$. Since the transit times may now vary over time, we have the following conditions for all $\xi \in [0, \alpha)$:

$$\ell_v(\phi) + \xi \cdot \ell'_v - \ell_u(\phi) - \xi \cdot \ell'_u > \tau_e(\ell_u(\phi) + \xi \cdot \ell'_u)) \qquad \text{for every } e \in E^*_\phi, \tag{5}$$

$$\ell_v(\phi) + \xi \cdot \ell'_v - \ell_u(\phi) - \xi \cdot \ell'_u < \tau_e(\ell_u(\phi) + \xi \cdot \ell'_u)) \qquad \text{for every } e \in E \setminus E'_\phi. \tag{6}$$

Furthermore, we need to ensure that the capacities of all active arcs and the network inflow rate do not change within the phase:

$$\nu_e(\ell_v(\phi)) = \nu_e(\ell_v(\phi) + \xi \cdot \ell'_v) \qquad \text{for every } e \in E'_\phi \text{ and all } \xi \in [0, \alpha). \tag{7}$$

$$r(\ell_s(\phi)) = r(\ell_s(\phi) + \xi \cdot \ell'_s) \qquad \text{for all } \xi \in [0, \alpha). \tag{8}$$

Finally, the speed ratios need to stay constant for all active arcs, i.e.,

$$\gamma_e(\ell_u(\phi)) = \gamma_e(\ell_u(\phi) + \xi \cdot \ell'_u) \qquad \text{for every } e \in E'_\phi \text{ and all } \xi \in [0, \alpha). \tag{9}$$

We call an $\alpha > 0$ *feasible* if it satisfies (5) to (9).

Lemma 5. *Given a restricted Nash flow over time f on $[0, \phi]$ then for right-constant capacities and speed limits there always exists a feasible $\alpha > 0$.*

Proof. By Lemma 4 we have that $\ell_v(\phi) - \ell_u(\phi) > \tau_e(\phi)$ for all $e \in E^*_\phi$ and $\ell_v(\phi) - \ell_u(\phi) < \tau_e(\phi)$ for all $e \in E \setminus E'_\phi$. Since τ_e is continuous there is an $\alpha_1 > 0$ such that (5) and (6) are satisfied for all $\xi \in [0, \alpha_1)$. Since ν_e, r and λ_e are right-constant so is γ_e, and hence, there is an $\alpha_2 > 0$ such that (7), (8) and (9) are fulfilled for all $\xi \in [0, \alpha_2)$. Clearly, $\alpha := \min\{\, \alpha_1, \alpha_2 \,\} > 0$ is feasible. □

For the maximal feasible α we call the interval $[\phi, \phi + \alpha)$ a *thin flow phase*.

Lemma 6. *An α-extension is a feasible flow over time and the extended ℓ-labels coincide with the earliest arrival times, i.e., they satisfy Equation (4) for all $\varphi \in [\phi, \phi + \alpha)$.*

The final step is to show that an α-extension is a restricted Nash flow over time on $[0, \phi + \alpha)$ and that we can continue this process up to ∞.

Theorem 3. *Given a restricted Nash flow over time $f = (f_e^+, f_e^-)_{e \in E}$ on $[0, \phi)$ in a time-varying network and a feasible $\alpha > 0$ then the α-extension is a restricted Nash flow over time on $[0, \phi + \alpha)$.*

Proof. Lemma 3 yields $F_e^+(\ell_u(\varphi)) = F_e^-(\ell_v(\varphi))$ for all $\varphi \in [0, \phi)$, so for $\xi \in [0, \alpha)$ it holds that

$$F_e^+(\ell_u(\phi+\xi)) = F_e^+(\ell_u(\phi)) + \frac{x'_e}{\ell'_u} \cdot \xi \cdot \ell'_u = F_e^-(\ell_v(\phi)) + \frac{x'_e}{\ell'_v} \cdot \xi \cdot \ell'_v = F_e^-(\ell_v(\phi+\xi)).$$

It follows again by Lemma 3 together with Lemma 6 that the α-extension is a restricted Nash flow over time on $[0, \phi + \alpha)$. □

Finally, we obtain our main result:

Theorem 4. *There exists a Nash flow over time in every time-varying network with right-constant speed limits, capacities and network inflow rates.*

Proof. The process starts with the empty flow over time, i.e., a restricted Nash flow over time for $[0, 0)$. We apply Theorem 3 with a maximal feasible α. If one of the α is unbounded we are done. Otherwise, we obtain a sequence $(f_i)_{i \in \mathbb{N}}$, where f_i is a restricted Nash flow over time for $[0, \phi_i)$, with a strictly increasing sequence $(\phi_i)_{i \in \mathbb{N}}$. In the case that this sequence has a finite limit, say $\phi_\infty < \infty$, we define a restricted Nash flow over time f^∞ for $[0, \phi_\infty)$ by using the point-wise limit of the x- and ℓ-labels, which exists due to monotonicity and boundedness of these functions. Note that there are only finitely many different thin flows, and therefore, the derivatives x' and ℓ' are bounded. Then the process can be restarted from this limit point. This so called *transfinite induction* argument works as follows: Let $\mathcal{P}_G$ be the set of all particles $\phi \in \mathbb{R}_{\geq 0}$ for which there exists a restricted Nash flow over time on $[0, \phi)$ constructed as described above. The set $\mathcal{P}_G$ cannot have a maximal element because the corresponding Nash flow over time could be extended by using Theorem 3. But $\mathcal{P}_G$ cannot have an upper bound either since the limit of any convergent sequence would be contained in this set. Therefore, there exists an unbounded increasing sequence $(\phi_i)_{i=1}^{\infty} \in \mathcal{P}_G$. As a restricted Nash flow over time on $[0, \phi_{i+1}]$ contains a restricted Nash flow over time on $[0, \phi_i]$ we can assume that there exists a sequence of *nested* restricted Nash flow over time. Hence, we can construct a Nash flow over time f on $[0, \infty)$ by taking the point-wise limit of the x- and ℓ-labels, completing the proof. □

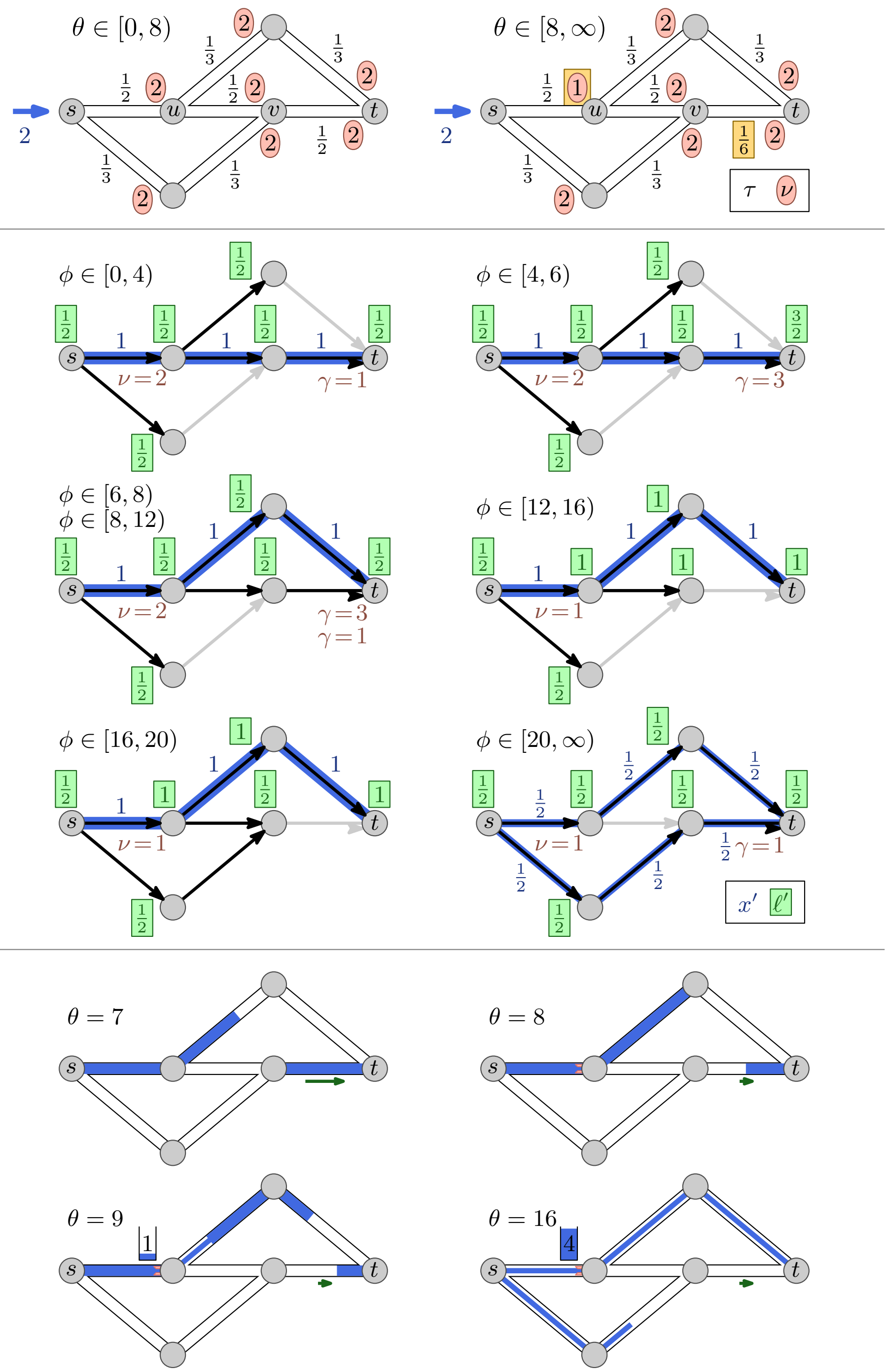

Fig. 5. A Nash flow over time with seven thin flow phases in a time-varying network.

Example. An example of a Nash flow over time in a time-varying network together with the corresponding thin flows is shown in Figure 5 on the next page. *On the top:* The network properties before time 8 (*left side*) and after time 8 (*right side*). *In the middle:* There are seven thin flow phases. Note that the third and forth phase (both depicted in the same network) are almost identical and only the speed ratio of arc vt changes, which does not influence the thin flow at all. *At the bottom:* Some key snapshots in time of the resulting Nash flow over time. The current speed limit λ_{vt} is visualized by the length of the green arrow and, for $\theta \geq 8$, the reduced capacity $\nu_{su}(\theta)$ is displayed by a red bottle-neck.

As displayed at the top the capacity of arc su drops from 2 to 1 at time 8 and, at the same time, the speed limit of arc vt decreases from $\frac{1}{2}$ to $\frac{1}{6}$. The first event for particle 4 is due to a change of the speed ratio leading to an increase of ℓ'_t. For particle 6, the top path becomes active and is taken by all following flow as particles on arc vt are still slowed down. For particle 8, the speed ratio at arc vt changes back to 1 but, as this arc is inactive, this does not change anything. Particle 12 is the first to experience the reduced capacity on arc su. The corresponding queue of this arc increases until the bottom path becomes active. This happens in two steps: first only the path up to node v becomes active for $\phi = 16$, and finally, the complete path is active from $\phi = 20$ onwards.

6 Conclusion and Open Problems

In this paper, we extended the base model that was introduced by Koch and Skutella, to networks which capacities and speed limits that changes over time. We showed that all central results, namely the existence of dynamic equilibria and their underlying structures in form of thin flow with resetting, can be transfered to this new model. With these new insights it is possible to model more general traffic scenarios in which the network properties are time-dependent. In particular, the flooding evacuation scenario, which was mentioned in the introduction, could not be modeled (not even approximately) in the base model.

There are still a lot of open question concerning time-varying networks. For example, it would be interesting to consider other flows over time in this setting, such as earliest arrival flows or instantaneous dynamic equilibria (see [11]) and show their existence. Can the proof for the bound of the price of anarchy [6] be transfered to this model, or is it possible to construct an example where the price of anarchy is unbounded?

References

1. Bhaskar, U., Fleischer, L., Anshelevich, E.: A stackelberg strategy for routing flow over time. Games and Economic Behavior **92**, 232–247 (2015)
2. Cao, Z., Chen, B., Chen, X., Wang, C.: A network game of dynamic traffic. In: Proc. of the 2017 ACM Conf. on Economics and Comp. pp. 695–696 (2017)
3. Cominetti, R., Correa, J., Larré, O.: Existence and uniqueness of equilibria for flows over time. In: Intern. Colloquium on Automata, Languages, and Programming. pp. 552–563. Springer (2011)

4. Cominetti, R., Correa, J., Larré, O.: Dynamic equilibria in fluid queueing networks. Operations Research **63**(1), 21–34 (2015)
5. Cominetti, R., Correa, J., Olver, N.: Long term behavior of dynamic equilibria in fluid queuing networks. In: Integer Programming and Combinatorial Optimization. pp. 161–172. Springer (2017)
6. Correa, J., Cristi, A., Oosterwijk, T.: On the price of anarchy for flows over time. In: Proc. of the 2019 ACM Conf. on Economics and Comp. pp. 559–577 (2019)
7. Fleischer, L., Tardos, É.: Efficient continuous-time dynamic network flow algorithms. Operations Research Letters **23**(3-5), 71–80 (1998)
8. Ford, L.R., Fulkerson, D.R.: Constructing maximal dynamic flows from static flows. Operations research **6**, 419–433 (1958)
9. Ford, L.R., Fulkerson, D.R.: Flows in Networks. Princeton University Press (1962)
10. Gale, D.: Transient flows in networks. The Michigan Mathematical Journal **6**(1), 59–63 (1959)
11. Graf, L., Harks, T., Sering, L.: Dynamic flows with adaptive route choice. Mathematical Programming (2020)
12. Harks, T., Peis, B., Schmand, D., Tauer, B., Vargas Koch, L.: Competitive packet routing with priority lists. ACM Transactions on Econo. and Comp. **6**(1), 4 (2018)
13. Kakutani, S.: A generalization of brouwer's fixed point theorem. Duke Mathematical Journal **8**, 457–459 (1941)
14. Koch, R.: Routing Games over Time. Ph.D. thesis, Technische Universitt Berlin (2012). https://doi.org/10.14279/depositonce-3347
15. Koch, R., Skutella, M.: Nash equilibria and the price of anarchy for flows over time. In: Intern. Symp. on Algorithmic Game Theory. pp. 323–334. Springer (2009)
16. Koch, R., Skutella, M.: Nash equilibria and the price of anarchy for flows over time. Theory of Computing Systems **49**(1), 71–97 (2011)
17. Macko, M., Larson, K., Steskal, L.: Braesss paradox for flows over time. Theory of Computing Systems **53**(1), 86–106 (2013)
18. Minieka, E.: Maximal, lexicographic, and dynamic network flows. Operations Research **21**(2), 517–527 (1973)
19. Peis, B., Tauer, B., Timmermans, V., Vargas Koch, L.: Oligopolistic competitive packet routing. In: 18th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (2018)
20. Scarsini, M., Schröder, M., Tomala, T.: Dynamic atomic congestion games with seasonal flows. Operations Research **66**(2), 327–339 (2018)
21. Sering, L., Skutella, M.: Multi-source multi-sink Nash flows over time. In: 18th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems. vol. 65, pp. 12:1–12:20 (2018)
22. Sering, L., Vargas Koch, L.: Nash flows over time with spillback. In: Proc. of the Thirtieth Annual ACM-SIAM Symp. on Discrete Algo. pp. 935–945. SIAM (2019)
23. Skutella, M.: An introduction to network flows over time. In: Research trends in combinatorial optimization, pp. 451–482. Springer (2009)
24. Vickrey, W.S.: Congestion theory and transport investment. The American Economic Review **59**(2), 251–260 (1969)
25. Wardrop, J.G.: Some theoretical aspects of road traffic research. Proc. of the institution of civil engineers **1**(5), 767–768 (1952)
26. Wilkinson, W.L.: An algorithm for universal maximal dynamic flows in a network. Operations Research **19**(7), 1602–1612 (1971)
27. Yagar, S.: Dynamic traffic assignment by individual path minimization and queuing. Transp. Res. **5**(3), 179–196 (1971)

7 Appendix: Technical Proofs

Lemma 1. *For all $e \in E$ and almost all $\theta \in [0, \infty)$ we have:*

(i) The function $\theta \mapsto \theta + \tau_e(\theta)$ is strictly increasing.
(ii) The function τ_e is continuous and almost everywhere differentiable.
(iii) For almost all $\theta \in [0, \infty)$ we have $1 + \tau_e'(\theta) = \frac{\lambda_e(\theta)}{\lambda_e(\theta+\tau_e(\theta))}$.

Proof. (i) Consider two points in time $\theta_1 < \theta_2$, then $\tau \coloneqq \theta_1 - \theta_2 + \tau_e(\theta_1)$ is strictly smaller than $\tau_e(\theta_2)$ since

$$\int_{\theta_2}^{\theta_2+\tau} \lambda_e(\xi)\,\mathrm{d}\xi = \int_{\theta_2}^{\theta_1+\tau_e(\theta_1)} \lambda_e(\xi)\,\mathrm{d}\xi < \int_{\theta_1}^{\theta_1+\tau_e(\theta_1)} \lambda_e(\xi)\,\mathrm{d}\xi = 1,$$

where the strict inequality holds, since λ_e is always strictly positive. The last equality follows by the definition of $\tau_e(\theta_1)$. Hence, with the definition of $\tau_e(\theta_2)$ we have $\theta_1 + \tau_e(\theta_1) < \theta_2 + \tau_e(\theta_2)$.

(ii) Since $\theta \mapsto \theta + \tau_e(\theta)$ is monotone, Lebesgue's theorem for the differentiability of monotone functions implies that it is almost everywhere differentiable. The same is then true for τ_e. The continuity follows directly from the definition since λ_e is always strictly positive.

(iii) By the definition of $\tau_e(\theta)$ we have

$$\int_0^{\theta+\tau_e(\theta)} \lambda_e(\xi)\,\mathrm{d}\xi - \int_0^{\theta} \lambda_e(\xi)\,\mathrm{d}\xi = 1.$$

Taking the derivatives of both sides and using Lebesgue's differentiation theorem together with the chain rule, we obtain

$$\lambda_e(\theta + \tau_e(\theta)) \cdot (1 + \tau_e'(\theta)) - \lambda_e(\theta) = 0.$$

Since λ_e is always strictly positive, we get

$$1 + \tau_e'(\theta) = \frac{\lambda_e(\theta)}{\lambda_e(\theta + \tau_e(\theta))}.$$

□

Lemma 2. *For a feasible flow over time f it holds for all $e \in E$, $v \in V$ and $\theta \in [0, \infty)$ that:*

(i) $q_e(\theta) > 0 \quad \Leftrightarrow \quad z_e(\theta + \tau_e(\theta)) > 0$.
(ii) $z_e(\theta + \tau_e(\theta) + \xi) > 0$ for all $\xi \in [0, q_e(\theta))$.
(iii) $F_e^+(\theta) = F_e^-(T_e(\theta))$.
(iv) For $\theta_1 < \theta_2$ with $F_e^+(\theta_2) - F_e^+(\theta_1) = 0$ and $z_e(\theta_2 + \tau_e(\theta_2)) > 0$ we have $T_e(\theta_1) = T_e(\theta_2)$.
(v) The functions T_e are monotonically increasing.
(vi) The functions q_e and T_e are continuous and almost everywhere differentiable.

(vii) For almost all $\theta \in [0, \infty)$ we have

$$T_e'(\theta) = \begin{cases} \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} & \text{if } q_e(\theta) > 0, \\ \max\left\{ \gamma_e(\theta), \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} \right\} & \text{else.} \end{cases}$$

Proof. (i) This follows directly from the definition of the waiting time q_e.

(ii) By equation (3) we have that $f_e^-(\xi) \leq \nu_e(\xi)$ almost everywhere. Hence, we have by definition that $q_e(\theta)$ is the minimal value such that

$$\int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+q_e(\theta)} \nu_e(\xi)\,\mathrm{d}\xi = z_e(\theta + \tau_e(\theta)).$$

Thus, we obtain for $\xi \in [0, q_e(\theta))$ that

$$\begin{aligned} F_e^-(\theta + \tau_e(\theta) + \xi) - F_e^-(\theta + \tau_e(\theta)) &= \int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+\xi} f_e^-(\xi)\,\mathrm{d}\xi \\ &\leq \int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+\xi} \nu_e(\xi)\,\mathrm{d}\xi \\ &< z_e(\theta + \tau_e(\theta)) \\ &= F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta)). \end{aligned}$$

Or in short: $F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta) + \xi) > 0$ for $\xi \in [0, q_e(\theta))$. Since F_e^+ is non-decreasing we obtain for all $\xi \in [0, q_e(\theta))$ that

$$z_e(\theta+\tau_e(\theta)+\xi) = F_e^+(\theta+\xi) - F_e^-(\theta+\tau_e(\theta)+\xi) \geq F_e^+(\theta) - F_e^-(\theta+\tau_e(\theta)+\xi) > 0.$$

(iii) By (3) and (ii) we obtain for almost all $\xi \in [\theta + \tau_e(\theta), \theta + \tau_e(\theta) + q_e(\theta))$ that $f_e^-(\xi) = \nu_e(\xi)$. By the definition of q_e we have

$$\begin{aligned} F_e^-(\theta + \tau_e(\theta) + q_e(\theta)) - F_e^-(\theta + \tau_e(\theta)) &= \int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+q_e(\theta)} f_e^-(\xi)\,\mathrm{d}\xi \\ &= \int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+q_e(\theta)} \nu_e(\xi)\,\mathrm{d}\xi \\ &= z_e(\theta + \tau_e(\theta)) \\ &= F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta)). \end{aligned}$$

Hence, $F_e^-(T_e(\theta)) = F_e^+(\theta)$.

(iv) Since $F_e^+(\theta_1) = F_e^+(\theta_2)$ we obtain with the monotonicity of F_e^- together with Lemma 1 (i) that

$$\begin{aligned} z_e(\xi + \tau_e(\xi)) &= F_e^+(\xi) - F_e^-(\xi + \tau_e(\xi)) \\ &\geq F_e^+(\theta_2) - F_e^-(\theta_2 + \tau_e(\theta_2)) = z_e(\theta_2 + \tau_e(\theta_2)) > 0, \end{aligned}$$

hence, (3) provides $f_e^-(\xi) = \nu_e(\xi)$ for almost all $\xi \in [\theta_1 + \tau_e(\theta_1), \theta_2 + \tau_e(\theta_2)]$. Thus, the definition of q_e implies that $q_e(\theta_1)$ equals

$$\begin{aligned} &\min\left\{ q \geq 0 \;\middle|\; \begin{aligned} &\int_{\theta_1+\tau_e(\theta_1)}^{\theta_2+\tau_e(\theta_2)} f_e^-(\xi)\,\mathrm{d}\xi + \int_{\theta_2+\tau_e(\theta_2)}^{\theta_1+\tau_e(\theta_1)+q} \nu_e(\xi)\,\mathrm{d}\xi \\ &\qquad = F_e^+(\theta_1) - F_e^-(\theta_1+\tau_e(\theta_1)) \end{aligned} \right\} \\ &= \min\left\{ p \geq 0 \;\middle|\; \int_{\theta_2+\tau_e(\theta_2)}^{\theta_2+\tau_e(\theta_2)+p} \nu_e(\xi)\,\mathrm{d}\xi = F_e^+(\theta_2) - F_e^-(\theta_2+\tau_e(\theta_2)) \right\} \\ &\qquad\qquad + \theta_2 + \tau_e(\theta_2) - \theta_1 - \tau_e(\theta_1) \\ &= q_e(\theta_2) + \theta_2 + \tau_e(\theta_2) - \theta_1 - \tau_e(\theta_1). \end{aligned}$$

Here, we substitute q by $p + \theta_2 + \tau_e(\theta_2) - \theta_1 - \tau_e(\theta_1)$ in order to obtain the first equation. Note that the condition $p \geq 0$ is always satisfied since the right hand side $F_e^+(\theta_2) - F_e^-(\theta_2 + \tau_e(\theta_2))$ equals $z_e(\theta_2 + \tau_e(\theta_2))$ and is therefore strictly positive by assumption. Hence, we obtain

$$T_e(\theta_1) = \theta_1 + \tau_e(\theta_1) + q_e(\theta_1) = \theta_2 + \tau_e(\theta_2) + q_e(\theta_2) = T_e(\theta_2).$$

(v) Considering two points in time $\theta_1 < \theta_2$, we show that $T_e(\theta_1) \leq T_e(\theta_2)$. Since F_e^+ is non-decreasing, (iii) implies that

$$F_e^-(T_e(\theta_2)) = F_e^+(\theta_2) \geq F_e^+(\theta_1) = F_e^-(T_e(\theta_1)). \tag{10}$$

If this holds with strict inequality, we obtain by monotonicity of F_e^- that $T_e(\theta_1) < T_e(\theta_2)$. If equation (10) holds with equality we have two cases. If $z_e(\theta_2 + \tau_e(\theta_2)) > 0$, (iv) states that $T_e(\theta_1) = T_e(\theta_2)$. If $z_e(\theta_2 + \tau_e(\theta_2)) = 0$, (ii) applied to θ_1 implies that $\xi := \theta_2 + \tau_e(\theta_2) - \theta_1 - \tau_e(\theta_1) \notin [0, q_e(\theta_1))$. Since $\xi \geq 0$ by Lemma 1 (i) we have $\xi \geq q_e(\theta_1)$, and thus,

$$T_e(\theta_2) \overset{\text{(i)}}{=} \theta_2 + \tau_e(\theta_2) \geq \theta_1 + \tau_e(\theta_1) + q_e(\theta_1) = T_e(\theta_1).$$

(vi) The continuity of q_e follows since ν_e is always strictly positive and z_e is continuous, as it is the difference of two continuous functions. Finally, T_e is continuous since it is the sum of three continuous functions.
By (v) the function T_e is non-decreasing for all $e \in E$, and hence, Lebesgue's theorem for the differentiability of monotone functions states that T_e is almost everywhere differentiable. Since $\theta \mapsto \theta + \tau_e(\theta)$ is monotone this also holds for τ_e since it is the difference of two almost everywhere differentiable functions. As a sum of almost everywhere differentiable functions, $q_e(\theta) = T_e(\theta) - \tau_e(\theta) - \theta$ has this property as well.

(vii) The definition of $q_e(\theta)$ states that

$$\int_0^{T_e(\theta)} \nu_e(\xi)\,\mathrm{d}\xi - \int_0^{\theta+\tau_e(\theta)} \nu_e(\xi)\,\mathrm{d}\xi = z_e(\theta+\tau_e(\theta)) = F_e^+(\theta) - F_e^-(\theta+\tau_e(\theta)).$$

Taking the derivative on both sides we obtain by using the chain rule that

$$\nu_e(T_e(\theta))\cdot T_e'(\theta)-\nu_e(\theta+\tau_e(\theta))\cdot(1+\tau_e'(\theta)) = f_e^+(\theta)-f_e^-(\theta+\tau_e(\theta))\cdot(1+\tau_e'(\theta)).$$

If $q_e(\theta) > 0$ we have by equation (3) that $f_e^-(\theta+\tau_e(\theta)) = \nu_e(\theta+\tau_e(\theta))$, and therefore, dividing by $\nu_e(T_e(\theta))$ (which is strictly positive by assumption) yields

$$T_e'(\theta) = \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))}.$$

For $q_e(\theta) = 0$ we have $f_e^-(\theta+\tau_e(\theta)) = \min\left\{ \frac{f_e^+(\theta)}{\gamma_e(\theta)}, \nu_e(\theta+\tau_e(\theta)) \right\}$ and $T_e(\theta) = \theta + \tau_e(\theta)$. Hence, dividing by $\nu_e(\theta+\tau_e(\theta)) = \nu_e(T_e(\theta))$ and using Lemma 1.(iii) provides

$$\begin{aligned} T_e'(\theta) &= \gamma_e(\theta) + \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} - \min\left\{ \frac{f_e^+(\theta)}{\gamma_e(\theta)}, \nu_e(T_e(\theta)) \right\} \cdot \frac{\gamma_e(\theta)}{\nu_e(T_e(\theta))} \\ &= \max\left\{ \gamma_e(\theta), \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} \right\}, \end{aligned}$$

which finishes the proof.

□

Theorem 2. *Consider an acyclic graph $G' = (V, E')$ with source s, sink t, capacities $\nu_e > 0$, speed ratios $\gamma_e > 0$ and a subset of arcs $E^* \subseteq E'$, as well as a network inflow $r > 0$. Furthermore, suppose that every node is reachable from s. Then there exists a thin flow $((x_e')_{e\in E}, (\ell_v')_{v\in V})$ with resetting on E^*.*

Proof. Let X be the compact, convex and non-empty set of all static s-t-flows of value 1 and let $\Gamma\colon X \to 2^X$ be defined by

$$x' \mapsto \{\, y' \in X \mid y_e' = 0 \text{ for all } e = uv \in E' \text{ with } \ell_v' < \rho_e(\ell_u', x_e') \,\},$$

where $(\ell_v')_{v\in V}$ are the node labels associated with x' uniquely defined by

$$\ell_v' = \begin{cases} \frac{1}{r} & \text{if } v = s, \\ \min\limits_{e=uv\in E'} \rho_e(\ell_u', x_e') & \text{if } v \in V \setminus \{\, s \,\}. \end{cases} \tag{11}$$

The existence of a fixed point of Γ is provided by Kakutani's fixed point theorem [13]

Theorem 5 (Kakutani's Fixed Point Theorem). *Let X be a compact, convex and non-empty subset of $\mathbb{R}^n$ and $\Gamma\colon X \to 2^X$, such that for every $x \in X$ the image $\Gamma(x)$ is non-empty and convex. Suppose the set $\{\, (x,y) \mid y \in \Gamma(x) \,\}$ is closed. Then there exists a fixed point $x^* \in X$ of Γ, i.e., $x^* \in \Gamma(x^*)$.*

All conditions are satisfied:

– The set $\Gamma(x')$ is non-empty, because there has to be at least one path P from s to t with $\ell_v' = \rho_e(\ell_u', x_e')$ for each arc e on P. If we set $y_e = 1$ for all arcs e on P and set every other value to 0 we obtain an element in $\Gamma(x')$.

- To see that $\Gamma(x')$ is convex, note that the arcs that can be used for sending flow, i.e., the ones satisfying $\ell'_v = \rho_e(\ell'_u, x'_e)$, are fixed within the set $\Gamma(x')$. Furthermore, every convex combination y of two elements $y^1, y^2 \in \Gamma(x')$ only uses arcs that are also used by y^1 or y^2.
- In order to show that the function graph $\{ (x', y') \mid y' \in \Gamma(x) \}$ is closed let $(x^n, y^n)_{n\in\mathbb{N}}$ be a sequence within this set, i.e., $y^n \in \Gamma(x^n)$. Since both sequences, $(x^n)_{n\in\mathbb{N}}$ and $(y^n)_{n\in\mathbb{N}}$, are contained in the compact set X they both have a limit x^* and y^* within X. Let $(\ell^n)_{n\in\mathbb{N}}$ be the sequence of associated node labels of (x^n) and ℓ^* the node label of x^*. Note that the mapping $x' \mapsto \ell'$ is continuous, and therefore, it holds that $\ell^* = \lim_{n\to\infty} \ell^n$. We prove that $y^* \in \Gamma(x^*)$. Suppose there is an arc $e = uv \in E'$ with $y^*_e > 0$ and $\ell^*_v < \rho_e(\ell^*_u, x^*_e)$. But since ρ_e is continuous there has to be an $n_0 \in \mathbb{N}$ such that $y^n_e > 0$ and $\ell^n_v < \rho_e(\ell^n_u, x^n_e)$ for all $n \geq n_0$, which is a contradiction. Hence, $\{ (x', y') \mid y' \in \Gamma(x) \}$ is closed.

Since all conditions for Kakutani's fixed point theorem are satisfied, there has to be a fixed point x^* of Γ. Let ℓ^* be the corresponding node labeling. We show that the pair (x^*, ℓ^*) satisfies the thin flow conditions. Equations (TF1) and (TF2) follow immediately by (11). For every arc $e = uv \in E'$ with $x^*_e > 0$ it holds that $\ell^*_v = \rho_e(\ell^*_u, x^*_e)$ since $x^* \in \Gamma(x^*)$, which shows Equation (TF3). Thus, (x^*, ℓ^*) forms a thin flow with resetting, which completes the proof. □

Lemma 6. *An α-extension is a feasible flow over time and the extended ℓ-labels coincide with the earliest arrival times, i.e., they satisfy Equation* (4) *for all* $\varphi \in [\phi, \phi + \alpha)$.

Proof. Flow conservation on nodes (2) holds since for all $\theta \in [\ell_v(\phi), \ell_v(\phi + \alpha))$ we have

$$\sum_{e\in\delta^+(v)} f^+_e(\theta) - \sum_{e\in\delta^-(v)} f^-_e(\theta) = \sum_{e\in\delta^+(v)} \frac{x'_e}{\ell'_v} - \sum_{e\in\delta^-(v)} \frac{x'_e}{\ell'_v} = \begin{cases} 0 & \text{if } v \in V \setminus \{ s, t \} \\ r(\ell_s(\phi)) \overset{(8)}{=} \theta & \text{if } v = s. \end{cases}$$

Next, we show that the feasibility condition (3) is satisfied. For this we first consider arcs e with $x'_e > 0$, which implies $e \in E'_\phi$. By (TF3) we have that $\ell'_v \geq \gamma_e(\ell_u(\phi)) \cdot \ell'_u$. Since γ is constant during the thin flow phase, so is τ', and therefore, we have for all $\xi \in [0, \alpha)$ that

$$\begin{aligned} \ell_v(\phi + \xi) &= \ell_v(\phi) + \xi \cdot \ell'_v \\ &\geq \ell_v(\phi) + \xi \cdot \gamma_e(\ell_u(\phi)) \cdot \ell'_u \\ &\geq \ell_u(\phi) + \tau_e(\ell_u(\phi)) + \xi \cdot (1 + \tau'_e(\ell_u(\phi)) \cdot \ell'_u \\ &= \ell_u(\phi + \xi) + \tau_e(\ell_u(\phi + \xi)). \end{aligned}$$

In other words, e stays active during the thin flow phase.

We consider the outflow rate at time $\theta + \tau_e(\theta)$ for $\theta \in [\ell_u(\phi), \ell_u(\phi + \alpha))$. In the case of $\theta + \tau_e(\theta) < \ell_v(\phi)$ the feasibility condition follows from prior phases.

Otherwise, $\theta + \tau_e(\theta) \in [\ell_v(\phi), \ell_v(\phi+\alpha))$, and therefore,

$$\begin{aligned} f_e^-(\theta+\tau_e(\theta)) &= \frac{x'_e}{\ell'_v} \overset{\text{(TF3)}}{=} \frac{x'_e}{\rho_e(\ell'_u, x'_e)} \\ &= \begin{cases} \min\left\{ \frac{x'_e}{\gamma_e(\ell_u(\phi))\cdot \ell'_u}, \nu_e(\ell_v(\phi)) \right\} & \text{if } e \in E'_\phi \setminus E^*_\phi, \\ \nu_e(\ell_v(\phi)) & \text{else,} \end{cases} \\ &= \begin{cases} \min\left\{ \frac{f_e^+(\theta)}{\gamma_e(\theta)}, \nu_e(\theta+\tau_e(\theta)) \right\} & \text{if } q_e(\theta) = 0, \\ \nu_e(\theta+\tau_e(\theta)) & \text{else.} \end{cases} \end{aligned}$$

In the case that $x'_e = 0$ we either have $\ell'_v = 0$, but then there is nothing to show since the interval $[\ell_v(\phi), \ell_v(\phi+\alpha))$ would be empty, or $\ell'_v > 0$, which means by (TF2) that either e is not active, or it is active but non-resetting. In both cases we have $q_e(\ell_u(\theta)) = 0$ and since $f_e^+(\ell_u(\theta)) = 0$ for all $\theta \in [\ell_u(\phi, \ell_u(\phi+\alpha))$ the queue stays empty during this phase. (3) follows since $f_e^-(\theta+\tau_e(\theta)) = \frac{x'_e}{\ell'_v} = 0 = f_e^+(\theta)$ holds for all $\theta \in [\ell_u(\phi, \ell_u(\phi+\alpha))$. Altogether, we showed that the α-extension is indeed a feasible flow over time.

It remains to show that Equation (4) holds, which implies that the extended ℓ-functions denote the earliest arrival times. First of all we have

$$\int_0^{\ell_s(\phi+\xi)} r(\xi)\,\mathrm{d}\xi = \phi + \int_{\ell_s(\phi)}^{\ell_s(\phi+\xi)} r(\xi)\,\mathrm{d}\xi = \phi + r(\ell_s(\phi))\cdot \ell'_s \cdot \xi = \phi+\xi.$$

Since r is always strictly positive, $\ell_s(\phi)$ is the minimal value with this property, which shows (4) for $v = s$. For $v \neq s$ we distinguish between three cases for every given arc $e = uv \in E$.

Case 1: $e \in E \setminus E'_\phi$.
Since α satisfies (6) it is satisfied for all $\xi \in [0, \alpha)$, and hence,

$$\begin{aligned} \ell_v(\phi+\xi) = \ell_v(\phi) + \xi \cdot \ell'_v &\overset{(6)}{\leq} \ell_u(\phi) + \xi\cdot \ell'_u + \tau_e(\ell_u(\phi) + \xi \cdot \ell'_u) \\ &\leq \ell_u(\phi+\xi) + \tau_e(\ell_u(\phi+\xi)) \leq T_e(\ell_u(\phi+\xi)). \end{aligned}$$

Case 2: $e \in E'_\phi \setminus E^*_\phi$ and $\gamma_e(\ell_u(\phi)) \cdot \ell'_u \geq \frac{x'_e}{\nu_e(\ell_v(\phi))}$.
Since e is active we have $\ell_v(\phi) = T_e(\ell_u(\phi)) = \ell_u(\phi) + \tau_e(\ell_u(\phi))$ and (TF2) implies $\ell'_v \leq \gamma_e(\ell_u(\phi))\cdot \ell'_u$. No queue builds up as $f_e^+(\ell_u(\phi+\xi)) = \frac{x'_e}{\ell'_u} \leq \nu_e(\ell_v(\phi))$, which means $z_e(\ell_u(\phi+\xi) + \tau_e(\ell_u(\phi))) = 0$ for all $\xi \in (0, \alpha]$. Combining these yields

$$\begin{aligned} \ell_v(\phi+\xi) &\overset{\text{(TF2)}}{\leq} \ell_v(\phi) + \xi \cdot \gamma_e(\ell_u(\phi)) \cdot \ell'_u \\ &= \ell_u(\phi) + \tau_e(\ell_u(\phi)) + \xi \cdot (1 + \tau'_e(\ell_u(\phi)) \cdot \ell'_u \\ &= \ell_u(\phi+\xi) + \tau_e(\ell_u(\phi+\xi)) \\ &= T_e(\ell_u(\phi+\xi)). \end{aligned}$$

Case 3: $e \in E^*_\phi$ or $\left(e \in E'_\phi \text{ and } \gamma_e(\ell_u(\phi)) \cdot \ell'_u < \frac{x'_e}{\nu_e(\ell_v(\phi))}\right)$.
Arc e is active ,i.e., $\ell_v(\phi) = T_e(\ell_u(\phi))$. We have $\rho_e(\ell'_u, x'_e) = \frac{x'_e}{\nu_e(\ell_v(\phi))}$, and hence, (TF2) implies $\ell'_v \leq \frac{x'_e}{\nu_e(\ell_v(\phi))}$. Lemma 2 (vii) yields

$$T'_e(\ell_u(\phi)) = \frac{f^+_e(\ell_u(\phi))}{\nu_e(\ell_v(\phi))} = \frac{x'_e}{\ell'_u \cdot \nu_e(\ell_v(\phi))}$$

since either $q_e(\ell_u(\phi)) > 0$ (if $e \in E^*$) or, in the case of $e \notin E^*_\phi$, we have

$$\frac{f^+_e(\ell_u(\phi))}{\nu_e(\ell_v(\phi))} = \frac{x'_e}{\ell'_u \cdot \nu_e(\ell_v(\phi))} > \gamma_e(\ell_u(\phi)).$$

Hence, for all $\xi \in (0, \alpha]$ we obtain

$$\begin{aligned}\ell_v(\phi + \xi) &\overset{\text{(TF2)}}{=} \ell_v(\phi) + \xi \cdot \ell'_v \leq \ell_v(\phi) + \xi \cdot \frac{x'_e}{\nu_e} \\ &= T_e(\ell_u(\phi)) + \xi \cdot T'_e(\ell_u(\phi)) \cdot \ell'_v = T_e(\ell_u(\phi + \xi)).\end{aligned}$$

This shows that there is no arc with an exit time earlier than the earliest arrival time, and therefore, the left hand side of (4) is always smaller or equal to the right hand side.

It remains to show that the equation holds with equality. For every node $v \in V \setminus \{ s \}$ there is at least one arc $e \in E'$ with $\ell'_v = \rho_e(\ell'_u, x'_e)$ in the thin flow due to (TF2). No matter if this arc belongs to Case 2 or Case 3 the corresponding equation holds with equality, which shows for all $\xi \in (0, \alpha]$ that

$$\ell_v(\phi + \xi) = \min_{e = uv \in E} T_e(\ell_u(\phi + \xi)).$$

This shows that (4) is also satisfied for $v \neq s$, which completes the proof. □

Lemma 7 (Differentiation rule for a minimum). *For every element e of a finite set E let $T_e \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function that is differentiable almost everywhere and let $\ell(\theta) := \min_{e \in E} T_e(\theta)$ for all $\theta \geq 0$. It holds that l is almost everywhere differentiable with*

$$\ell'(\theta) = \min_{e \in E'_\theta} T'_e(\theta) \tag{12}$$

for almost all $\theta \geq 0$ where $E'_\theta := \{ e \in E \mid \ell(\theta) = T_e(\theta) \}$.

Proof. Let $\phi \geq 0$ such that all T_e, for all $e \in E$, are differentiable, which is almost everywhere. Since all functions T_e are continuous at ϕ we have for sufficiently small $\varepsilon > 0$ that $\ell(\phi + \xi) = \min_{e \in E'_\phi} T_e(\phi + \xi)$ for all $\xi \in [\phi, \phi + \varepsilon]$. It follows that

$$\begin{aligned}\lim_{\xi \searrow 0} \frac{\ell(\phi + \xi) - \ell(\phi)}{\xi} &= \lim_{\xi \searrow 0} \min_{e \in E'_\phi} \frac{T_e(\phi + \xi) - \ell(\phi)}{\xi} \\ &= \min_{e \in E'_\phi} \lim_{\xi \searrow 0} \frac{T_e(\phi + \xi) - T_e(\phi)}{\xi} = \min_{e \in E'_\phi} T'_e(\phi).\end{aligned}$$

Note that every point ϕ where all T_e are differentiable, but for which the left derivative of ℓ does not coincide with the right derivative of ℓ, is a proper crossing of at least two T_e functions. Therefore, these points are isolated and form a null set. Hence, we have $\ell'(\phi) = \min_{e \in E'_\phi} T'_e(\phi)$ for almost all $\phi \in \mathbb{R}_{\geq 0}$. $\square$