Nash Flows over Time with Spillback*

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Abstract

Modeling traffic in road networks is a widely studied but challenging problem, especially under the assumption that drivers act selfishly. A common approach used in simulation software is the *deterministic queuing model*, for which the structure of dynamic equilibria has been studied extensively in the last couple of years. The basic idea is to model traffic by a continuous flow that travels over time from a source to a sink through a network, in which the arcs are endowed with transit times and capacities. Whenever the flow rate exceeds the capacity a queue builds up and the infinitesimally small flow particles wait in line in front of the bottleneck. Since the queues have no physical dimension, it was not possible, until now, to represent spillback in this model. This was a big drawback, since spillback can be regularly observed in real traffic situations and has a huge impact on travel times in highly congested regions. We extend the deterministic queuing model by introducing a storage capacity that bounds the total amount of flow on each arc. If an arc gets full, the inflow capacity is reduced to the current outflow rate, which can cause queues on previous arcs and blockages of intersections, i.e., spillback. We carry over the main results of the original model to our generalization and characterize dynamic equilibria, called *Nash flows over time*, by sequences of particular static flows, we call *spillback thin flows*. Furthermore, we give a constructive proof for the existence of dynamic equilibria, which suggests an algorithm for their computation. This solves an open problem stated by Koch and Skutella in 2010 [12].

Keywords network congestion, Nash equilibrium, dynamic routing game, spillback, flow over time, deterministic queuing model

Topics Algorithmic aspects of networks, Game theory and mechanism design, Scheduling and resource allocation

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1 Introduction

Urban population is rapidly growing worldwide and so is the number of vehicles in metropolitan areas. To get control of this rising traffic volume intelligent traffic planning is of central importance. In particular it is essential to solve many of the major traffic problems in today's cities, e.g., air and noise pollution and long travel times. In other words, a well planned traffic does not only increase the quality of life for traffic users but also benefits the economy and environment. Improved navigation systems and the availability of massive amounts of traveling data give a huge opportunity to optimize the infrastructure for the growing demand. This draws the attention to more realistic mathematical traffic models and algorithmic approaches for the interplay of individual road users. Unfortunately, on the one hand, realistic models used in simulations are mathematically poorly understood and, on the other hand, theoretically precise models that are mathematically well-analyzed are very simplified. Our contribution is to extend the theoretical state of the art model by adding a crucial component: *spillback*. This effect can be observed in daily traffic situations, e.g., on a highway, where a bottleneck causes a long traffic jam that blocks exits upstream, or during rush hour in a big city where a crossing is impassable due to the congestion of an intersecting road. It is no surprise that spillback is of great interest for traffic planners and that it is a core feature of recent traffic simulation tools. In other words, introducing spillback is an important step towards closing the gap between mathematical models and simulations.

Flows over time. As there is a huge number of interacting agents in daily traffic, we do not concentrate on single entities but consider traffic streams instead. For this scenario, in which infinitesimally small agents travel through a network over time, *flows over time* are an excellent mathematical description. While the game theoretical perspective of this problem is still in its infancy, the optimization perspective has already been studied for more than half a century. In 1958, Ford and Fulkerson [7] introduced a time-dependent flow model, in which flow travels over time through a network from a source s to a sink t. Every arc of the network is equipped with a capacity, which limits the rate of flow using that arc, and a transit time specifying the time needed to traverse it. This model is widely analyzed and there are several algorithms solving different optimization problems. Ford and Fulkerson presented an algorithm for the maximum flow over time problem, i.e., sending as much flow as possible from s to t within a given time horizon. A natural extension is to search for flows over time that maximizes the flow amount reaching the sink for every point in time simultaneously, a so called *earliest arrival flows*. In 1959, Gale [8] proved their existence in an s-t-network and the first algorithm was presented by Minieka [13] and Wilkinson [17]. All these problems were first considered from a discrete time perspective and only in 1996, Fleischer and Tardos [6] showed that all the results and algorithms carry over to the continuous time model, which has become the conventional perspective by now. For a very nice introduction into the whole field we refer to the survey of Skutella [16].

Dynamic equilibria. Meanwhile flows over time were considered from a decentralized, game theoretical perspective in the transportation science community; see, e.g., the book of Ran and Boyce [15] and the article about spillback of Daganzo [5]. In accurate traffic scenarios it is reasonable to expect the participants (particles) to act selfishly, i.e., to minimize their arrival times. The actual traffic is then represented by a dynamic equilibrium, i.e., a state where no particle can reach the destination quicker by changing its route. In this paper we consider the *deterministic queuing model* to describe the arc dynamics, which is also used in the simulation software MATSim [10]. In this competitive flow over time setting it is possible that the inflow rate exceeds the capacity for some arc, which causes a queue to build up in front of the exit. Therefore, the actual travel time of an arc consists of the transit time plus the queue waiting time. Koch and Skutella [12] characterized the structure of dynamic equilibria, called *Nash flows over time*, and showed that they consist of a number of phases, within which the in- and outflow rates of each arc are constant. Each phase is characterized by a particular static flow together with node labels, named *thin flows with resetting*. Cominetti, Correa, and Larré [2] showed the existence and uniqueness of these thin flows with resetting, from which a Nash flow over time can be constructed. In [3] they extended the existence result to networks with more general inflow rate functions and to a multi-commodity setting. In 2017 Comenetti, Correa, and Olver [4] examined the long term behavior of queues and were able to bound their lengths whenever the network capacity is sufficiently large. Finally, Bhaskar, Fleischer and Anshelevich [1] analyzed different prices of anarchy in this model.

Our contribution. In the deterministic queuing model the queues do not have physical dimensions and can in principle be arbitrarily large. Thus, spillback cannot occur, which is a huge drawback when considering real world scenarios. Our contribution is to extend this model such that the total amount of flow on an arc, and thus the queue length, can be bounded. Whenever an arc is *full* the inflow rate cannot exceed the outflow rate anymore. In words of traffic: if a road is full no new vehicle can enter the street before another vehicle leaves. If more flow aims to use a full arc, it has to queue up on a previous arc, i.e., we have *spillback*. We generalize the concept of thin flows to this spillback setting by introducing an additional node label, which we call *spillback factor*. We show that, similar to the original model, the derivatives of every Nash flow over time with spillback form a *spillback thin flow*. In reverse it is possible to compute a Nash flow over time by extending the flow over time step by step via spillback thin flows.

Outline. In Section 2 we motivate the spillback model via road properties and give an illustrative example to emphasize the importance of spillback. Section 3 introduces the basic notations and concepts of the flow dynamics. In Section 4 we define Nash flows over time and spillback thin flows and show their structural connection. Section 5 is dedicated to the construction and computation of Nash flows over time via spillback thin flows. Finally, in Section 6, we give a brief conclusion and outlook on further interesting questions. Due to space constraints and readability we moved the majority of the proofs and some technical lemmas to the appendix.

2 From Roads to Arcs

Street model. In order to get an appropriate model imagine a street with a number of lanes w, a length ℓ, and a speed limit v_1; see on the left of Figure 1. We assume the street ends in a crossing, which leads to an exit speed limit denoted by v_2. The number of cars entering or leaving the street per time unit is denoted by $f^+(\theta)$ and $f^-(\theta)$ respectively. Whenever some cars cannot leave the road immediately a traffic jam builds up at the end of the road. Its length at time θ is denoted by $j(\theta)$ and if it equals ℓ, the street is full. A new car can only enter the street if there is enough space on a lane at this moment. After entering it drives along the street with velocity v_1 until it reaches either the end of the street or the end of the traffic jam. In the latter case, it stays in the stop-and-go traffic until it reaches the end of the street. Whenever the following street is full the outflow is restricted, therefore the stop-and-go speed is reduced even further leading to longer traffic jams. This is what we call spillback.

Arc model. To describe this situation mathematically we consider a directed graph. Hereby, each arc corresponds to a street segment and every node to a crossing. In order to make the street dynamics easier to handle we transform the street properties in the following way (see on the right of Figure 1): Each arc is equipped with an inflow capacity ν^+ corresponding to $w \cdot v_1$, a free flow transit time τ corresponding to ℓ/v_1, a storage capacity σ corresponding to $\ell \cdot w/(\text{car length})$ and

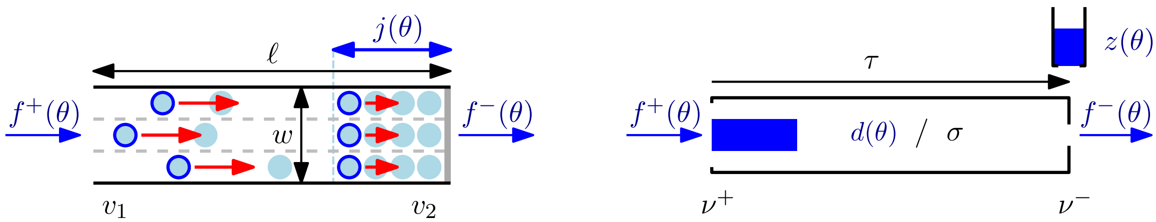

Figure 1: *Left:* Model of cars using a street with w lanes, length ℓ, speed limit v_1, and exit speed v_2. *Right:* Simplification we use within this paper to model a street. Cars are modeled as a flow and the traffic jam is replaced by a point queue. The link attributes correspond to the physical properties of the street.

an outflow capacity ν^- corresponding to $v_2 \cdot w$. Instead of considering discrete cars we look at a continuous flow over time that is described by an inflow rate function f^+ and an outflow rate function f^-. In this flow model the queue does not have a physical length, i.e., each flow particle first traverses the arc in τ time and if the *point queue* $z(\theta)$ is positive the particle lines up. It is not hard to see that the in- and outflow rate are the same whether the queues have physical length or not. The total amount of flow on an arc, consisting of traversing particles and the flow in the queue, is denoted by $d(\theta)$ and can never exceed the storage capacity σ.

Introductory example. To illustrate the importance of spillback we present two examples of Nash flows over time. We consider the same network in both cases except for a different outflow capacity on arc e_2. In the first case, depicted on the left side of Figure 2, suppose that the outflow capacity is $\nu^-_{e_2} = 1$. Since the unique shortest path in the network uses arc e_2 all flow particles use this path until time 4. At this point in time particles located at node v that decide to use e_2 will be at the end of this arc at time 5 and will experience a queue of length 6. Hence the total travel time along e_2 is 7, which equals the transit time of e_3. Thus, the flow splits up: a rate of 2 takes arc e_3 and the remaining flow of rate 1 chooses arc e_2 nevertheless. Since the inflow rate of arc e_2 is now 1, and therefore equal to the outflow rate, the queue length stays constant at 6 and the total travel time from s to t remains constant at 8 for all times.

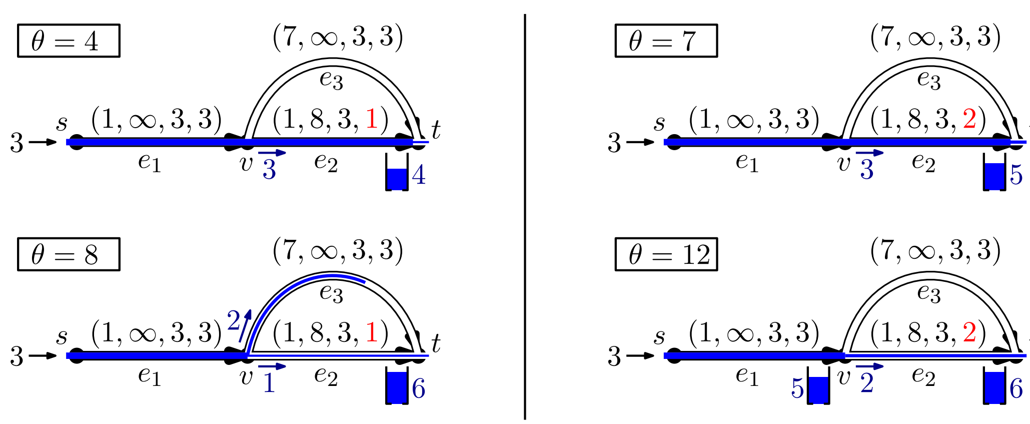

Figure 2: The arc properties are displayed in the following order $(\tau, \sigma, \nu^+, \nu^-)$. *Left:* We have $\nu^-_{e_2} = 1$, and therefore the ring road e_3 is used from time 4 onward. *Right:* We widen the capacity to $\nu^-_{e_2} = 2$. Now it is faster for all particles to stick to the main road e_2 even after this arc gets full at time 7.

For the second case, depicted on the right of Figure 2, we consider the same network except for the outflow capacity $\nu^-_{e_2} = 2$, which changes the situation drastically. As before, in the beginning the path along e_2 is the unique shortest route, and thus all particles take it. At time 7 the arc gets

full because the amount of flow in the queue equals 5 and 3 additional flow units are traversing the arc at this moment. From this time onward the inflow rate into e_2 is restricted to the outflow rate, namely 2. Note that for particles located at v at time 7 the travel time along e_2 equals 4: one time unit for traversing the arc and a waiting time of 3 (at time 8 the amount of flow in the queue is 6 and the particles leave the queue with a rate of 2). Hence, for these particles e_2 is still faster than e_3 and since the queue will never become longer than 6 all later particles will stick to the bottom route. Since e_2 is full, they spill back and queue up on arc e_1 from time 7 onward. It follows that the total travel time from s to t will rise unbounded for later particles.
This example points out that the storage capacity might have a huge influence on the dynamic equilibrium and it shows that this can even lead to counter-intuitive dynamics, since widening the capacity on e_2 leads to a longer travel time for later particles.

3 Spillback Model

In this section we introduce the dynamic queuing model with spillback, i.e., we specify the properties of the network and the flow dynamics on the arcs. Note that the model is a generalization of the model introduced in [12] and the structure of this article follows the lines of [2, 3, 12].
Throughout this paper we consider a directed graph $G = (V, E)$ with transit times $\tau_e \geq 0$, outflow capacities $\nu_e^- > 0$, inflow capacities $\nu_e^+ > 0$ and storage capacities $\sigma_e \in (0, \infty]$ on every arc $e \in E$. Furthermore, there are two distinguished nodes, a source $s \in V$ with an inflow rate $r \geq 0$ and a sink $t \in V$. We assume that every node is reachable from s and that there is no directed cycle with zero transit time. In order to ensure that traversing flow alone can never fill up the storage of an arc e we require that $\sigma_e > \nu_e^+ \cdot \tau_e$. We assume that $\sigma_e = \infty$ and $\nu_e^+ > r$ for all $e \in \delta^+(s)$ and $\delta^-(s) = \emptyset$. In other words, spillback can never reach the source, and thus the network inflow is never throttled. This is without loss of generality, because we can ensure the requirements by adding a super source s^* and a new arc $e^* = s^*s$ with $\tau_{e^*} = 0$, $\sigma_{e^*} = \infty$, $\nu_{e^*}^- = r$, and $\nu_{e^*}^+ = r + 1$. It is possible to disable the inflow restriction for some arc $e = uv$ by choosing an inflow capacity ν_e^+ larger than the potential total inflow into u, namely $\sum_{e \in \delta^-(u)} \nu_e^-$.

Flows over time. The time-depending flows considered in this paper are specified by $f = (f_e^+, f_e^-)_{e \in E}$, where $f_e^+ \colon [0, \infty) \to [0, \infty)$ and $f_e^- \colon [0, \infty) \to [0, \infty)$ are locally integrable and bounded functions for every arc e. The function f_e^+ describes the *inflow rate* and f_e^- the *outflow rate* of arc e for every given point in time $\theta \in [0, \infty)$. The cumulative in- and outflow functions are defined as follows:

$$F_e^+(\theta) := \int_0^\theta f_e^+(\xi)\,\mathrm{d}\xi \qquad \text{and} \qquad F_e^-(\theta) := \int_0^\theta f_e^-(\xi)\,\mathrm{d}\xi.$$

Due to technical reasons we define $F_e^+(\theta) = F_e^-(\theta) = 0$ for $\theta < 0$. Note that it follows immediately that F_e^+ and F_e^- are monotonically increasing and Lipschitz continuous. We say $f = (f_e^+, f_e^-)_{e \in E}$ is a *flow over time* if it *conserves flow* at every node $v \in V \setminus \{ t \}$, i.e., if for all $\theta \in [0, \infty)$ the following equality holds

$$\sum_{e \in \delta^+(v)} f_e^+(\theta) - \sum_{e \in \delta^-(v)} f_e^-(\theta) = \begin{cases} 0 & \text{if } v \neq s \\ r & \text{if } v = s. \end{cases}$$

Queues. We consider a bottleneck given by the outflow capacity at the end of every arc. If the flow rate that wants to leave e exceeds the outflow capacity, a queue builds up, which we imagine as a point queue at the head of the arc, as depicted in Figure 2. The amount of flow in the queue at time θ is given by $z_e(\theta) := F_e^+(\theta - \tau_e) - F_e^-(\theta)$. Note that flow always leaves the queue as fast as possible, which is indirectly implied by the feasibility conditions below.

Full arcs and flow bounds. The *arc load* is the total amount of flow on an arc e and is given by $d_e(\theta) := F_e^+(\theta) - F_e^-(\theta)$. It is the sum of the flow traversing the arc and the flow in the queue at a point in time θ. We say the arc is *full* at time θ if $d_e(\theta) = \sigma_e$. For technical reasons we also say an arc e is full if $d_e(\theta) > \sigma_e$ even though we show in Lemma 1 that this can never happen for a feasible flow over time (see below). The *inflow bound* of an arc e is defined as follows

$$b_e^+(\theta) := \begin{cases} \min\{\, f_e^-(\theta), \nu_e^+ \,\} & \text{if } e \text{ is full at time } \theta \\ \nu_e^+ & \text{else,} \end{cases}$$

and the *push rate* of an arc e is defined by

$$b_e^-(\theta) := \begin{cases} 0 & \text{if } \theta < \tau_e \\ \nu_e^- & \text{if } \theta \geq \tau_e \text{ and } z_e(\theta) > 0 \\ \min\{\, f_e^+(\theta - \tau_e), \nu_e^- \,\} & \text{if } \theta \geq \tau_e \text{ and } z_e(\theta) \leq 0. \end{cases}$$

The value $b_e^-(\theta)$ describes the rate, with which the flow leaves arc e at time θ if it is not restricted by any spillback. Obviously, this is an upper bound on the actual outflow rate $f_e^+(\theta)$, which is captured by the fair allocation condition below. Due to spillback it is possible that the actual outflow rate of some arc e is strictly less then the push rate, i.e., $f_e^-(\theta) < b_e^-(\theta)$. In this case we call e *throttled* at time θ.

Feasibility. A flow over time f is *feasible* if it satisfies the following four conditions:

- *Inflow condition:* We have $f_e^+(\theta) \leq b_e^+(\theta)$ for all θ and every arc $e \in E$.
- *Fair allocation condition:* For every node v at time θ there is a $c_v(\theta) \in (0, 1]$ such that for all incoming arcs $e \in \delta^-(v)$ we have $f_e^-(\theta) = \min\{\, b_e^-(\theta), \nu_e^- \cdot c_v(\theta) \,\}$.
- *No slack condition:* For every node v we have that if there is an incoming arc that is throttled at time θ, then there has to be at least one outgoing arc $e \in \delta^+(v)$ with $f_e^+(\theta) = b_e^+(\theta)$.
- *No deadlock condition:* For every point in time θ the set of full arcs is cycle free.

Intuitively, the fair allocation condition ensures that the total node inflow is shared among the incoming throttled arcs proportionally to there outflow capacities (zipper principle) and the no slack conditions makes sure that no arc is throttled causeless. In a feasible flow over time the outflow rates should never exceed the outflow capacity, queues should never become negative, and the arcs should not get overfull. All this follows from the feasibility conditions above.

Lemma 1. *A feasible flow over time satisfies the following conditions for all θ and every arc e:*

(i) Outflow capacity condition: $f_e^-(\theta) \leq \nu_e^-$.

(ii) Non-deficit condition: $z_e(\theta) \geq 0$.

(iii) Storage condition: $d_e(\theta) \leq \sigma_e$.

The proof for this is straightforward; see B.1.

Spillback factor. For every node v we call the *maximal* value $c_v(\theta) \in (0, 1]$ that fulfills the fair allocation condition the *spillback factor* for node v at time θ.

Travel and arrival times. Given a network and a feasible flow over time, an important question is at which time a sample flow particle starting at s at time θ can reach a node $v \in V$. First, we consider the *waiting time* in the queue for a particle entering an arc e at time θ, which is given by

$$q_e(\theta) \coloneqq \min \left\{ q \geq 0 \;\middle|\; \int_{\theta+\tau_e}^{\theta+\tau_e+q} f_e^-(\xi) \; \mathrm{d}\xi = z_e(\theta+\tau_e) \right\}.$$

To show that the set on the right hand side is never empty, and thus $q_e(\theta)$ is well-defined, we prove that there is a network-wide lower bound on the outflow rate of arcs with positive queues.

Lemma 2. *For a given network there is an $\varepsilon > 0$ such that for every arc e with $z_e(\theta) > 0$ we have $f_e^-(\theta) \geq \varepsilon$, and therefore the waiting time function q_e is well-defined.*

Note that there can be a long chain of full arcs behind e reducing the outflow rate significantly. But due to the no deadlock condition and the no slack condition there has to be an arc at the end where the inflow capacity is exhausted. Hence, using the fair allocation condition it is possible to choose ε only depending on the capacities and the total number of arcs. For a formal proof see B.2.
A particle entering an arc e at time θ first traverses the arc in τ_e time, then waits in the queue for $q_e(\theta)$ time units before it leaves the arc at the *exit time*

$$T_e(\theta) \coloneqq \theta + \tau_e + q_e(\theta).$$

We denote the time a particle starting at time θ needs to traverse a path $P = (e_1, \dots, e_k)$ by

$$T_P(\theta) \coloneqq T_{e_k} \circ \ldots \circ T_{e_1}(\theta).$$

The *earliest arrival time function* $\ell_v \colon [0,\infty) \to [0,\infty)$ maps a time θ to the earliest time a sample particle, starting at time θ at source s, can reach node v. This value is given by

$$\ell_v(\theta) \coloneqq \min_{P \in \mathcal{P}_v} T_P(\theta),$$

where $\mathcal{P}_v$ denotes the set of all s-v-paths. Furthermore, these ℓ-labels are characterized by the following dynamic Bellman's equations, which is shown by Lemma 12 in A.5.

$$\ell_v(\theta) = \begin{cases} \theta & \text{if } v = s \\ \min\limits_{e=uv \in \delta^-(v)} T_e(\ell_u(\theta)) & \text{if } v \neq s. \end{cases} \tag{1}$$

It is worth noting that all these q-, T-, and ℓ-functions are Lipschitz continuous, and thus almost everywhere differentiable due to Rademacher's theorem [14]. Furthermore, the waiting time functions q_e do not decrease faster than with slope -1 and T_e and ℓ_e are monotonically increasing. These and further technical properties are collected in Lemmas 10 and 11; see A.3 and A.4.

Active, resetting and spillback arcs. For every point in time θ we define the following classes of arcs. We say an arc is *active* for θ if it attains the minimum in (1), i.e., the set of active arcs is

$$E'_\theta = \{ \, e = uv \in E \mid \ell_v(\theta) = T_e(\ell_u(\theta)) \, \}.$$

The subgraph $G'_\theta \coloneqq (V, E'_\theta)$, is called *current shortest paths network*. It follows from (1) that G'_θ is acyclic and connected, which will be important later on and is proven in Lemma 13; see A.6.
We call the set of arcs on which the particle entering at time θ would experience a queue *resetting arcs* and arcs that are full when the particle would arrive there are called *spillback arcs*. We denote them by

$$E^*_\theta \coloneqq \{ \, e = uv \in E \mid q_e(\ell_u(\theta)) > 0 \, \} \quad \text{and} \quad \bar{E}_\theta \coloneqq \{ \, e = uv \in E \mid d_e(\ell_u(\theta)) = \sigma_e \, \}, \text{ respectively.}$$

4 Nash Flows Over Time and Spillback Thin Flows

In this section we define a dynamic equilibrium, called *Nash flow over time*, for the spillback model and we show, as a central structural result, that the strategy of every particle can be described by a particular static flow, which we call *spillback thin flow*.

Nash flows over time. A feasible flow over time $f = (f_e^+, f_e^-)_{e\in E}$ is a *Nash flow over time* if it satisfies the *Nash flow condition*, i.e., for almost all $\theta \in [0,\infty)$ and all arcs $e = uv$ we have

$$f_e^+(\theta) > 0 \quad \Rightarrow \quad \theta \in \left\{\, \ell_u(\vartheta) \in [0,\infty) \;\middle|\; e \in E'_\vartheta \,\right\}.$$

Remark 3. *A game theoretical Nash equilibrium is a state such that no player can improve by choosing an alternative strategy. Since for every particle starting in s at time θ the earliest possible arrival time $\ell_t(\theta)$ is realized, there is no improving move from the perspective of a single particle.*

Underlying static flow. In Lemma 14 (see A.7) we show besides some other characterizations that a flow over time f is a Nash flow over time if and only if $F_e^+(\ell_u(\theta)) = F_e^-(\ell_v(\theta))$ for all arcs $e = uv$ and all times θ. This motivates to define the *underlying static flow* for every point in time θ:

$$x_e(\theta) := F_e^+(\ell_u(\theta)) = F_e^-(\ell_v(\theta)).$$

It is easy to verify that for a fixed time θ this is indeed a static s-t-flow of flow value $r \cdot \theta$ and that x_e as a function is monotonically increasing and Lipschitz continuous.
Applying Rademacher's theorem [14] to x_e and ℓ_v we obtain derivatives $x'_e(\theta)$ and $\ell'_v(\theta)$ almost everywhere. Note that it is possible to reconstruct the Nash flow over time by these derivative functions, since $x'_e(\theta) = f_e^+(\ell_u(\theta)) \cdot \ell'_u(\theta) = f_e^-(\ell_v(\theta)) \cdot \ell'_v(\theta)$. Furthermore, $x'(\theta)$ forms a static s-t-flow of value r and can be seen as the strategy of the flow entering the network at time θ. In other words, these derivative functions characterize a Nash flow over time and it turns out that they have a very particular structure, which we call *spillback thin flows*. This is a generalization of *thin flows with resetting* introduced in [12].

Spillback thin flows. Consider an acyclic directed graph $G' = (V, E')$ with a source s and a sink t where all nodes are reachable from s. Every arc e is equipped with an outflow capacity $\nu_e^- > 0$ and an inflow bound $b_e^+ > 0$. Additionally, we are given a subset of arcs $E^* \subseteq E'$. A static s-t-flow x' of value r (which does not need to obey the capacities) together with two node labelings $\ell'_v \geq 0$ and $c_v \in (0,1]$ is a *spillback thin flow* with resetting on E^* if it fulfills the following equations:

$$\ell'_s = \frac{1}{c_s} \tag{TF1}$$

$$\ell'_v = \min_{e=uv\in E'} \rho_e(\ell'_u, x'_e, c_v) \qquad \text{for all } v \in V\setminus\{\, s \,\} \tag{TF2}$$

$$\ell'_v = \rho_e(\ell'_u, x'_e, c_v) \qquad \text{for all } e = uv \in E' \text{ with } x'_e > 0 \tag{TF3}$$

$$\ell'_v \geq \max_{e=vw\in E'} \frac{x'_e}{b_e^+} \qquad \text{for all } v \in V \tag{TF4}$$

$$\ell'_v = \max_{e=vw\in E'} \frac{x'_e}{b_e^+} \qquad \text{for all } v \in V \text{ with } c_v < 1, \tag{TF5}$$

where

$$\rho_e(\ell'_u, x'_e, c_v) := \begin{cases} \frac{x'_e}{c_v\cdot\nu_e^-} & \text{if } e = uv \in E^* \\ \max\left\{\, \ell'_u, \frac{x'_e}{c_v\cdot\nu_e^-} \,\right\} & \text{if } e = uv \in E'\setminus E^*. \end{cases}$$

The next theorem describes the relation between spillback thin flows and Nash flows over time.

Theorem 4 (Derivatives of underlying static flows are spillback thin flows)**.** *The derivatives $x'_e(\theta)$ and $\ell'_v(\theta)$ of a Nash flow over time together with the spillback factors $c_v(\ell_v(\theta))$ form a spillback thin flow for almost all $\theta \in [0, \infty)$ on the current shortest paths network $G'_\theta = (V, E'_\theta)$ with resetting on the arcs with queue E^*_θ and inflow bounds $b^+_e(\ell_u(\theta))$.*

The proof of Equations (TF1) to (TF3) mainly consists of a case distinction in order to determine $T'_e(\ell_u(\theta)) \cdot \ell'_u(\theta)$. Equation (TF4) follows immediately from the inflow condition and Equation (TF5) follows from the no slack and the fair allocation condition. See B.3 for a detailed proof.

5 Computation of Nash flows Over Time with Spillback

In this section we show how to construct a Nash flow over time with spillback for a given network using spillback thin flows. The key idea is to start with the empty flow over time and to extend it step by step. For this we first show that for all acyclic networks $G' = (V, E')$ with arbitrary capacities, outflow bounds, and resetting arcs E^* there always exists a spillback thin flow.

Theorem 5 (Existence of spillback thin flows)**.** *Consider an acyclic network $G' = (V, E')$ with source s and sink t, such that each node is reachable from s. Furthermore, let $(\nu^-_e)_{e\in E'}$ be outflow capacities, $(b^+_e)_{e\in E'}$ be inflow bounds, and $E^* \subseteq E'$ be a set arcs. Then there exists a spillback thin flow (x', ℓ', c) with resetting on E^*.*

To prove the existence of a spillback thin flow we follow the lines of the proof of Theorem 4 in [3] and adapt it to the spillback model. We start by converting Equations (TF1) to (TF5) to a continuous function Γ with a compact and convex domain K and consider a variational inequality that asks for a solution $x \in K$ such that $(y - x)^t\Gamma(x) \geq 0$ for all $y \in K$. By a variation of Brouwer's fixed-point theorem [9, 11] such a solution exists. In the next step we show that every solution x that satisfies some additional requirements fulfills the non-linear complementary problem: $x \cdot \Gamma(x) = 0$ and $\Gamma(x) \geq 0$. With this and the structure of the spillback thin flow constraints we can show that x corresponds to a spillback thin flow. For details we refer to B.4.
Note that a spillback thin flow can be computed with a mixed integer program. In addition to the flow constraints and the conditions (TF1) to (TF5) we have to add binary decider variables w_e for every non-resetting but active arc, y_e for every active arc and z_v for every node, where

$$
\begin{aligned}
w_e = 1 \quad &\Leftrightarrow \quad \ell_u \geq x'_e/(\nu^-_e \cdot c_v), &&\text{and thus } \rho_e(x'_e, \ell'_u, c_v) = \ell'_u,\\
y_e = 1 \quad &\Leftrightarrow \quad x'_e = 0, &&\text{and thus (TF3) does not apply,}\\
z_v = 1 \quad &\Leftrightarrow \quad c_v = 1, &&\text{and thus (TF5) does not apply.}
\end{aligned}
$$

Since there is no objective function every feasible solution is already a spillback thin flow.

α-Extensions. Let $\phi \geq 0$ be a fixed point in time. A feasible flow over time with piece-wise constant and right-continuous functions (f^+, f^-) is a *restricted Nash flow over time* on $[0, \phi)$ if it is a Nash flow over time for the inflow function $r_\phi(\theta) = r \cdot \mathbb{1}_{[0,\phi]}$, where $\mathbb{1}$ is the indicator function. In a Nash flow over time the FIFO principle holds, i.e., no particle entering the network at time $\theta \geq \phi$ can influence any particle that has entered the network before time ϕ. Thus, all the previous results carry over to restricted Nash flows over time. The earliest arrival times $\ell_u(\phi)$ can be determined by taking the left-sided limits, which provide us with the current shortest paths network $G'_\phi = (V, E'_\phi)$ and the resetting arcs E^*_ϕ. It is further possible to determine the spillback arcs $\bar{E}_\phi$ and the inflow bounds $b^+_e(\ell_u(\phi))$. More details can be found in Lemma 15 in A.8.

By Theorem 5 we can obtain a spillback thin flow (x', ℓ', c) on the current shortest paths network G'_ϕ with resetting on E^*_ϕ and inflow bounds $(b^+_e(\ell_u(\phi)))_{e\in E'_\phi}$. We set $x'_e := 0$ for all $e \in E\backslash E'_\phi$ and extend the following functions linearly for some $\alpha > 0$:

$$\ell_v(\theta) := \ell_v(\phi) + (\theta - \phi) \cdot \ell'_v \quad \text{and} \quad x_e(\theta) := x_e(\phi) + (\theta - \phi) \cdot x'_e \qquad \text{for } \theta \in [\phi, \phi + \alpha).$$

Furthermore, the inflow and outflow function of every arc $e = uv \in E$ are extended by

$$f^+_e(\theta) := x'_e/\ell'_u \quad \text{for } \theta \in [\ell_u(\phi), \ell_u(\phi + \alpha)) \quad \text{and} \quad f^-_e(\theta) := x'_e/\ell'_v \quad \text{for } \theta \in [\ell_v(\phi), \ell_v(\phi + \alpha)),$$

and the cumulative flow functions F^+_e and F^-_e are extended accordingly. Note that $\ell'_u = 0$ implies that the interval $[\ell_u(\phi), \ell_u(\phi+\alpha))$ is empty, and therefore f^+_e is not changed in this case. The same is true for f^-_e if $\ell'_v = 0$. We call the family of extended flow functions $(f^+_e, f^-_e)_{e\in E}$ an *α-extension.*

Extension step size. In the following we present some necessary boundaries on α, which we later show to be sufficient for the α-extension to form a restricted Nash flow over time on $[0, \phi+\alpha)$. First, the waiting times cannot become negative or, in other words, flow cannot traverse an arc faster than τ_e and, second, non active arcs can get active, and therefore open alternative routes for particles. These two properties are captured by the following two conditions on α:

$$\ell_v(\phi) - \ell_u(\phi) + \alpha(\ell'_v - \ell'_u) \geq \tau_e \qquad \text{for all } e = uv \in E^*_\phi \tag{2}$$

$$\ell_v(\phi) - \ell_u(\phi) + \alpha(\ell'_v - \ell'_u) \leq \tau_e \qquad \text{for all } e = uv \in E\backslash E'_\phi. \tag{3}$$

In addition, the inflow bounds of the spillback arcs need to be constant within one extension phase,

$$b^+_e(\ell_u(\phi) + \theta \cdot \ell'_u) = b^+_e(\ell_u(\phi)) \qquad \text{for all } e = uv \in \bar{E}_\phi \text{ and all } \theta \in [0, \alpha). \tag{4}$$

Finally, non-spillback arcs can become full, which changes the spillback thin flow. So the following condition makes sure that the total amount of flow on a non-spillback arc stays strictly under the storage capacity within the extension phase:

$$F^+_e(\ell_u(\phi + \xi)) - F^-_e(\ell_u(\phi + \xi)) < \sigma_e \qquad \text{for all } e = uv \in E'_\phi\backslash\bar{E}_\phi \text{ and all } \xi \in [0, \alpha). \tag{5}$$

Note that F^-_e needs not to be linear on $[\ell_u(\phi), \ell_u(\phi + \alpha))$.
We call $\alpha > 0$ *feasible* if it satisfies Equations (2) to (5) and such an α always exists, which we show in Lemma 16; see A.9. For the maximal feasible α we call $[\phi, \phi + \alpha)$ *thin flow phase.*

Computing Nash flows over time. The next theorem shows that it is possible to extend a restricted Nash flow over time with spillback step by step using α-extensions. We cannot hope for a polynomial time algorithm, since there are examples with exponential number of thin flow phases [4], which means that the output is of exponential size. Nevertheless, the constructive nature of the α-extensions leads to an algorithm which might be output-polynomial depending on the computation complexity of a spillback thin flow, which is still open.

Theorem 6 (α-Extensions are restricted Nash flows over time)**.** *Given a restricted Nash flow over time on $[0, \phi)$ and a feasible $\alpha > 0$, the α-extension is a restricted Nash flow over time on $[0, \phi+\alpha)$. Furthermore, the extended ℓ-functions are indeed the earliest arrival times and the extended x-functions describe the underlying static flow for all $\theta \in [0, \phi + \alpha)$.*

In the proof we first show that the α-extension is a feasible flow over time, where the fair allocation condition follows from (TF2) and (TF3), the inflow condition from (TF4), and the no slack condition from (TF5). To show that the extended ℓ-labels correspond to the earliest arrival times we do a quite technical case distinction, but the Nash flow condition follows immediately. The proof can be found in full detail in B.5.
Theorem 7 finally shows the existence of Nash flows over time in the spillback setting.

Theorem 7. *There exists a Nash flow over time with spillback.*

Proof. The empty flow over time is a restricted Nash flow over time for the empty set $[0,0)$. For a given restricted Nash flow over time f_i on $[0,\phi_i)$ we choose a maximal feasible $\alpha_i > 0$, which exists due to Lemma 16, and extend f_i with Theorem 6 to a restricted Nash flow over time f_{i+1} on $[0,\phi_{i+1})$, where $\phi_{i+1} = \phi_i + \alpha_i$. This leads to a strictly increasing sequence $(\phi_i)_{i\in\mathbb{N}}$. Suppose this sequence has a finite limit $\phi_\infty = \lim_{i\to\infty}\phi_i < \infty$. In this case we define a restricted Nash flow over time f^∞ for $[0,\phi_\infty)$ by using the point-wise limits of the x- and ℓ-functions. Note that the functions remain Lipschitz continuous, and therefore the process can be continued from this limit point. Since this enables us to always extend the Nash flow over time, there cannot be an upper bound on the length of the extension interval because the smallest upper bound would correspond to a limit point, which we can extend again. □

Experiments suggest that the number of phases is finite, but we were not able to prove this.

Example. In Figure 3 we display the spillback thin flows of the introductory example.

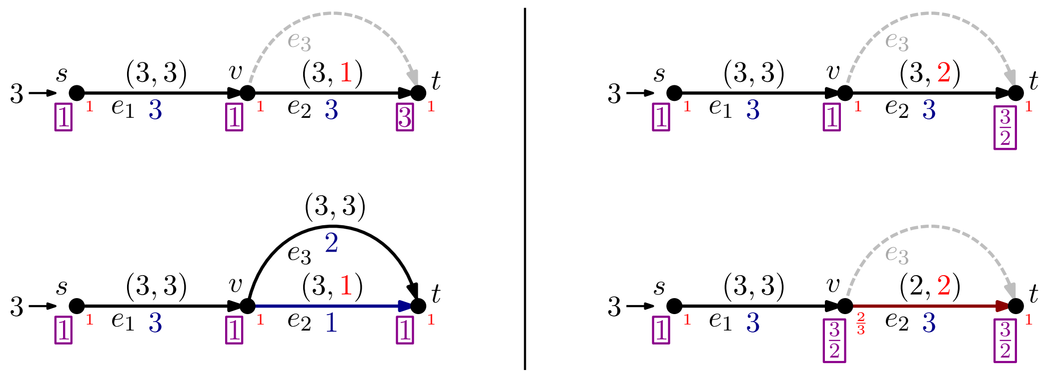

Figure 3: The arc properties are displayed in the following order (b^+,ν^-). Furthermore, the numbers under the arcs are x', the numbers in the boxes are ℓ', and the small numbers next to the nodes are c. Dashed arcs are non-active. *Left:* For $\nu^-_{e_2} = 1$ there are two phases, and thus two spillback thin flows. In the second phase e_3 becomes active and e_2 is resetting. *Right:* For $\nu^-_{e_2} = 2$ there are also two phases. In the second phase e_2 becomes full and is therefore a spillback arc with $b^+_{e_2} = 2$. Since $c_v = 2/3 < 1$ we have spillback.

6 Conclusion and Outlook

We introduced the spillback model and showed that dynamic equilibria can be constructed by using a sequence of static flows. It is worth noting that this spillback model is a generalization of the deterministic queuing model studied in [2, 3, 12], since the spillback feature can be disabled by setting the storage capacities to infinity and the inflow rate large enough; see Remark 17. For the original model Cominetti et. al. showed in [4] that if the inflow rate r does not exceed the capacity of a minimal cut, then the lengths of all queues are bounded. This is not the case in the spillback model, as the example on the right of Figure 2 shows. But it is still possible that there exists a phase in the computation of a Nash flow over time that lasts indefinitely. It remains open to characterize such long term behavior and to give a bound on the number of phases. Furthermore, it is a challenging open problem in the original as well as in the spillback model to compute a thin flow efficiently or to show any hardness results. Since Nash flows over time are intended to describe traffic situations, a model with multiple origin-destination-pairs would be a huge step. This is a very difficult problem since the earliest arrival times differ for every commodity.

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A Additional Lemmas

A.1 Arc Saturation

Lemma 8. *If an arc e is full at time θ, this implies $z_e(\theta) > 0$.*

Proof. The lemma holds true due to the following equations:

$$z_e(\theta) = F_e^+(\theta - \tau_e) - F_e^-(\theta) \geq F_e^+(\theta) - \nu_e^+ \cdot \tau_e - F_e^-(\theta) = \sigma_e - \nu_e^+ \tau_e > 0.$$

The first equation holds by definition of z_e, the inequality follows since ν_e^+ is an upper bound on the inflow f_e^+ due to the inflow condition. The next equality follows since the arc is full and thus $\sigma_e = d_e(\theta)$, while the last inequality holds by the requirement on the storage capacity. □

A.2 Uniqueness of Outflow Rates

Lemma 9. *Fix a point in time θ. Given a node $v \in V \setminus \{ s, t \}$ with inflow rates $f_e^+(\theta)$ for all $e \in \delta^+(v)$ such that*

$$\Omega_v(\theta) := \sum_{e \in \delta^+(v)} f_e^+(\theta) > 0$$

and given inflow rates $f_e^+(\theta - \tau_e)$ and queues $z_e^+(\theta)$ for all incoming arcs $e \in \delta^-(v)$. Suppose the potential maximal inflow of v is greater or equal to the total outflow at v, i.e.,

$$\sum_{e \in \delta^-(v)} b_e^-(\theta) \geq \Omega_v(\theta)$$

then there exist a unique family of outflow rates $(f_e^-(\theta))_{e \in \delta^-(v)}$ and a spillback factor $c_v(\theta)$ that fulfill flow conservation and the fair allocation condition at v.

Proof. **Existence:** For each $a \in [0, 1]$ and every arc $e \in \delta^-(v)$ let

$$g_e^a := \min \{ b_e^-(\theta), a \cdot \nu_e^- \} .$$

The function $H : [0, 1] \to \mathbb{R}_{\geq 0}$ with $a \mapsto H(a) := \sum_{e \in \delta^-(v)} g_e^a$ is continuous and increasing with $H(0) = 0 < \Omega_v(\theta) \leq \sum_{e \in \delta^-(v)} b_e^-(\theta) = H(1)$. By the intermediate value theorem there exists a $c \in (0, 1]$ with $H(c) = \Omega_v(\theta)$ and the set of values that fulfill this equation is compact. Let $c_v(\theta)$ be the maximal element of this set. We set $f_e^-(\theta) := g_e^{c_v(\theta)}$ for all arcs e. Clearly, flow is conserved at v and by definition of $g_e^{c_v(\theta)}$ the fair allocation condition is satisfied.
Uniqueness: Let $(g'_e)_{e \in \delta^-(v)}$ be also a solution with c' as spillback factor, i.e., $g'_e = g_e^{c'}$. If there is no throttled arc we have $g'_e = b_e^-(\theta) = g_e^{c_v(\theta)}$ for all arcs $e \in \delta^-(v)$ and $c' = c_v(\theta) = 1$. Since the total outflow rate $H(a)$ is strictly increasing within $\{ a' \in [0, 1] \mid H(a') < H(1) \}$, flow conservation gives us $H(c') = \Omega_v(\theta) = H(c_v(\theta))$, and therefore $c' = c_v(\theta)$, which implies $g'_e = g_e^{c_v(\theta)}$ for all arcs $e \in \delta^-(v)$. □

A.3 Technical Properties

Lemma 10. *For a feasible flow over time the following statements hold true for all arcs $e \in E$ at all times $\theta \geq 0$.*

(i) $F_e^+(\theta) = F_e^-(T_e(\theta))$.

(ii) $q_e(\theta) > 0 \quad \Leftrightarrow \quad z_e(\theta + \tau_e) > 0$.

(iii) For $\theta_1 < \theta_2$ *with* $F_e^+(\theta_2) - F_e^+(\theta_1) = 0$, *and* $z_e(\theta_2 + \tau_e) > 0$ *we have* $T_e(\theta_1) = T_e(\theta_2)$.

(iv) If $f_e^-(T_e(\theta)) = 0$ *we have* $F_e^+(\theta + q_e(\theta)) - F_e^+(\theta) = 0$.

(v) For the push rate function it holds that

$$b_e^-(T_e(\theta)) = \begin{cases} \nu_e^- & \text{if } F_e^+(\theta + q_e(\theta)) - F_e^+(\theta) > 0 \\ \min\{\, f_e^+(T_e(\theta) - \tau_e), \nu_e^- \,\} & \text{else.} \end{cases}$$

(vi) We have $z_e(\theta + \tau_e + \xi) > 0$ *for all* $\xi \in [0, q_e(\theta))$.

(vii) The function T_e *is monotonically increasing.*

(viii) The functions q_e *and* T_e *are Lipschitz continuous.*

Proof. (i) By definition of F_e^- and z_e we get the following alternative description of q_e.

$$\begin{aligned} q_e(\theta) &= \min\left\{\, q \geq 0 \;\middle|\; F_e^-(\theta + \tau_e + q) - F_e^-(\theta + \tau_e) = F_e^+(\theta) - F_e^-(\theta + \tau_e) \,\right\} \\ &= \min\left\{\, q \geq 0 \;\middle|\; F_e^-(\theta + \tau_e + q) = F_e^+(\theta) \,\right\} \end{aligned}$$

Thus, $q_e(\theta)$ is the first point in time such that

$$F_e^+(\theta) = F_e^-(\theta + \tau_e + q_e(\theta)) = F_e^-(T_e(\theta)).$$

(ii) This follows directly by the definition of q_e and since f_e^- is bounded by ν_e^-.

(iii) Intuitively this holds true since for a particle entering the end of a queue, the entering time does not influence the time to leave the queue, if no other particle enters the queue in between the two times and if the queue does not empty out. Formally, this follows with

$$\begin{aligned} q_e(\theta_1) &= \min\left\{ q \geq 0 \;\middle|\; \int_{\theta_1+\tau_e}^{\theta_1+\tau_e+q} f_e^-(\xi)\, d\xi = F_e^+(\theta_1) - F_e^-(\theta_1 + \tau_e) \right\} \\ &= \min\left\{ q \geq 0 \;\middle|\; \int_{\theta_2+\tau_e}^{\theta_1+\tau_e+q} f_e^-(\xi)\, d\xi + \int_{\theta_1+\tau_e}^{\theta_2+\tau_e} f_e^-(\xi)\, d\xi = F_e^+(\theta_2) - F_e^-(\theta_1 + \tau_e) \right\} \\ &= \min\left\{ p = q - \theta_2 + \theta_1 \geq 0 \;\middle|\; \int_{\theta_2+\tau_e}^{\theta_2+\tau_e+p} f_e^-(\xi)\, d\xi = F_e^+(\theta_2) - F_e^-(\theta_2 + \tau_e) \right\} + \theta_2 - \theta_1 \\ &= q_e(\theta_2) + \theta_2 - \theta_1. \end{aligned}$$

Thus, $\theta_1 + \tau_e + q_e(\theta_1) = \theta_2 + \tau_e + q_e(\theta_2)$.

(iv) For $q_e(\theta) = 0$, the claim follows immediately. So assume $q_e(\theta) > 0$ and to show the contraposition of the claim suppose $F_e^+(\theta + q_e(\theta)) - F_e^+(\theta) > 0$. We obtain

$$z_e(T_e(\theta)) = F_e^+(\theta + q_e(\theta)) - F_e^-(T_e(\theta)) \overset{(i)}{=} F_e^+(\theta + q_e(\theta)) - F_e^+(\theta) > 0 \tag{6}$$

and by Lemma 2 it follows that $f_e^-(T_e(\theta)) > \varepsilon$.

(v) Equation (6) states that $z_e(T_e(\theta)) > 0$ if and only if $F_e^+(\theta + q_e(\theta)) - F_e^+(\theta) > 0$ and thus the claim follows by definition of $b_e^-(\theta)$.

(vi) Since there is nothing to show for $q_e(\theta) = 0$ assume $q_e(\theta) > 0$. By definition, $q_e(\theta)$ is the minimal positive number such that $F_e^-(\theta+\tau_e+q_e(\theta)) = F_e^+(\theta)$, and therefore $F_e^+(\theta)-F_e^-(\theta+\tau_e+\xi) > 0$ for $\xi \in [0, q_e(\theta))$. Since F_e^+ is monotonically increasing we have for all $\xi \in [0, q_e(\theta))$ that

$$z_e(\theta + \tau_e + \xi) = F_e^+(\theta + \xi) - F_e(\theta + \tau_e + \xi) \geq F_e^+(\theta) - F_e(\theta + \tau_e + \xi) > 0.$$

(vii) Consider two points in time $\theta_1 < \theta_2$. Since F_e^+ is monotonically increasing, we get with (i) that

$$F_e^-(T_e(\theta_2)) = F_e^+(\theta_2) \geq F_e^+(\theta_1) = F_e^-(T_e(\theta_1)). \tag{7}$$

If (7) holds with strict inequality, we obtain $T_e(\theta_1) < T_e(\theta_2)$, since F_e^- is monotonically increasing as well. In the case that (7) holds with equality we distinguish two cases. If $z_e(\theta_2 + \tau_e) > 0$, (iii) states that $T_e(\theta_1) = T_e(\theta_2)$. In the case that $z_e(\theta_2 + \tau_e) = 0$ it follows by (ii) applied to θ_1 that $\xi := \theta_2 - \theta_1 \notin [0, q_e(\theta_1))$ and thus with (ii)

$$T_e(\theta_2) = \theta_2 + \tau_e + q_e(\theta_2) = \theta_2 + \tau_e \geq \theta_1 + \tau_e + q_e(\theta_1) = T_e(\theta_1).$$

(viii) To show that $q_e(\theta)$ is Lipschitz continuous, we need to identify a constant L such that for all $\theta_1 < \theta_2$ we have

$$|q_e(\theta_2) - q_e(\theta_1)| \leq L \cdot |\theta_2 - \theta_1| .$$

First we observe that $q_e(\theta_1)$ cannot be a lot larger than $q_e(\theta_2)$, since F_e^+ and F_e^- are monotonically increasing. By monotonicity of T_e, see (vii), it holds that

$$q_e(\theta_2) - q_e(\theta_1) = T_e(\theta_2) - \theta_2 - \tau_e - (T_e(\theta_1) - \theta_1 - \tau_e) \geq -(\theta_2 - \theta_1) .$$

So we consider the more interesting case that $q_e(\theta_2) > q_e(\theta_1)$ and give an upper bound on $q_e(\theta_2) - q_e(\theta_1)$.

Since the maximal inflow of arc $e = uv$ is bounded by ν_e^+ we get an upper bound for the total amount of inflow between θ_1 and θ_2:

$$F_e^+(\theta_2) - F_e^+(\theta_1) \leq \nu_e^+ (\theta_2 - \theta_1) .$$

In the following we show that we can restrict ourselves to the interval preceding θ_2 with a positive queue on arc e. If this is already the case for the given interval $[\theta_1, \theta_2]$, we stick to it. If not we reduce the interval to $[\theta_0, \theta_2] \subseteq [\theta_1, \theta_2]$ such that there is a positive queue on the arc during all of the interval. By having this property we can guarantee a constant outflow of at least ε, which allow us to define a global L. In the following we present the technical details. If there exists a $\theta \in [\theta_1, \theta_2]$ with $q_e(\theta) = z_e(T_e(\theta)) = 0$, we replace θ_1 by $\theta_0 := \max\{\, \theta \in [\theta_1, \theta_2] \mid q_e(\theta) = 0 \,\}$. With Lemma 10 (vi) we get that $z_e(\theta) > 0$ for all

$\theta \in (T_e(\theta_0), T_e(\theta_2))$ and, more importantly, $q_e(\theta_2) - q_e(\theta_1) \leq q_e(\theta_2) - q_e(\theta_0)$. The same is obviously true if $z_e(T_e(\theta)) > 0$ for all $\theta \in [\theta_1, \theta_2]$ and we choose $\theta_0 \coloneqq \theta_1$.

From Lemma 2 we get that the outflow f_e^- is bounded from below by ε as long as there is a queue on e.

It follows that

$$F_e^-(T_e(\theta_2)) - F_e^-(T_e(\theta_0)) \geq \varepsilon \cdot (T_e(\theta_2) - T_e(\theta_0)) = \varepsilon \cdot (\theta_2 + q_e(\theta_2) - \theta_0 - q_e(\theta_0)).$$

With the upper equations and Lemma 10 (i) we get

$$\begin{aligned} \nu_e^+ (\theta_2 - \theta_1) &\geq F_e^+(\theta_2) - F_e^+(\theta_1) = F_e^-(T_e(\theta_2)) - F_e^-(T_e(\theta_1)) \geq F_e^-(T_e(\theta_2)) - F_e^-(T_e(\theta_0)) \\ &\geq \varepsilon \cdot (\theta_2 + q_e(\theta_2) - \theta_0 - q_e(\theta_0)) \geq \varepsilon \cdot (\theta_2 - \theta_1) + \varepsilon \cdot (q_e(\theta_2) - q_e(\theta_0)). \end{aligned}$$

Finally, we have

$$q_e(\theta_2) - q_e(\theta_1) \leq q_e(\theta_2) - q_e(\theta_0) \leq \left(\frac{\nu_e^+ - \varepsilon}{\varepsilon}\right)(\theta_2 - \theta_1),$$

and therefore we can choose $L \coloneqq \max\left\{\frac{\nu_e^+ - \varepsilon}{\varepsilon}, 1\right\}$, which shows that q_e is Lipschitz continuous and that T_e is Lipschitz continuous follows immediately from the definition.

□

A.4 Derivatives of Waiting Times

From the property that $q_e(\theta)$ is Lipschitz continuous it follows by Rademacher's theorem [14] that it is almost everywhere differentiable.

Lemma 11. *For almost all θ the following is true:*

$$q_e'(\theta) = \begin{cases} \frac{f_e^+(\theta)}{f_e^-(T_e(\theta))} - 1 & \text{if } f_e^-(T_e(\theta)) > 0 \\ -1 & \text{else if } z_e(\theta + \tau_e) > 0 \\ 0 & \text{else.} \end{cases}$$

Proof. By definition of $q_e(\theta)$ we have

$$F_e^-(T_e(\theta)) - F_e^-(\theta + \tau_e) = \int_{\theta+\tau_e}^{\theta+\tau_e+q_e(\theta)} f_e^-(\xi)\ \mathrm{d}\xi = z_e(\theta).$$

The functions F_e^-, z_e, and q_e are Lipschitz continuous, and therefore almost everywhere differentiable. Taking the derivative on both sides, we obtain

$$f_e^-(T_e(\theta)) \cdot (1 + q_e'(\theta)) - f_e^-(\theta + \tau_e) = z_e'(\theta).$$

If $f_e^-(T_e(\theta)) > 0$ we get together with $z_e'(\theta) = f_e^+(\theta) - f_e^-(\theta + \tau_e)$ that

$$q_e'(\theta) = \frac{f_e^+(\theta)}{f_e^-(T_e(\theta))} - 1.$$

In the case of $f_e^-(T_e(\theta)) = 0$ and $z_e(\theta+\tau_e) > 0$ we have by Lemma 10 (iv) that $F_e^+(\theta+\xi) - F_e^+(\theta) = 0$ for all $\xi \in [0, q_e(\theta)) \neq \emptyset$, and therefore $T_e(\theta) = T_e(\theta+\xi)$ by Lemma 10 (iii). It follows that

$$q_e(\theta+\xi) = T_e(\theta+\xi) - \theta - \xi - \tau_e = T_e(\theta) - \theta - \tau_e - \xi = q_e(\theta) - \xi.$$

Hence, the right derivative of q_e at θ equals -1. Therefore, either q is not differentiable at θ or $q_e'(\theta) = -1$.
Finally, we consider the case $f_e^-(T_e(\theta)) = 0$ and $z_e(\theta+\tau_e) = 0$. Since q_e is non negative and $q_e(\theta) = 0$ by Lemma 10 (ii), θ is a local minimum of q_e. Thus, either q_e is not differentiable at θ or $q_e'(\theta) = 0$. □

A.5 Bellman's Equations

Lemma 12. *For a node $v \in V$ it holds that*

$$\ell_v(\theta) = \begin{cases} \theta & \text{if } v = s \\ \min\limits_{e=uv\in\delta^-(v)} T_e(\ell_u(\theta)) & \text{if } v \neq s. \end{cases}$$

Proof. For $v = s$ we take the empty path and obtain $\ell_v(\theta) \leq T_{()}(\theta) = \theta$. Furthermore, $T_P(\theta) \geq \theta$ for all paths P, and therefore we have equality. Now consider $v \neq s$. For every path $P_u := (e_1, \ldots, e_k) \in \mathcal{P}_u$ we can consider $P_v := (e_1, \ldots, e_k, e) \in \mathcal{P}_v$ for $e = uv$, and therefore by the monotonicity of T_e it follows that

$$\ell_v(\theta) \leq T_{P_v}(\theta) = T_e(T_{P_u}(\theta)) \leq T_e(\ell_u(\theta)).$$

It remains to show that there is an arc such that this holds with equality. For this let $P_v := (e_1, \ldots, e_k = uv) \in \mathcal{P}_v$ now be the proper s-v-path that attains the minimum in the definition of $\ell_v(\theta)$, i.e., $\ell_v(\theta) = T_P(\theta)$, and let $P_u := (e_1', \ldots, e_{k'}') \in \mathcal{P}_u$ be a path with $\ell_u(\theta) = T_{P_u}(\theta)$. By considering $P_v' := (e_1', \ldots, e_{k'}', e_k) \in \mathcal{P}_v$ and $P_u' := (e_1, \ldots, e_{k-1}) \in \mathcal{P}_u$ we get by the definition of the earliest arrival times and the monotonicity of T_{e_k} that

$$\ell_v(\theta) \leq T_{P_v'}(\theta) = T_{e_k}(\ell_u(\theta)) \leq T_{e_k}(T_{P_u'}(\theta)) = \ell_v(\theta),$$

and therefore equality. □

A.6 Current Shortest Paths Networks are Acyclic

Lemma 13. *The graph $G_\theta' = (V, E_\theta')$ is acyclic and every node is reachable from s.*

Proof. We start by proving that $G_\theta' = (V, E_\theta')$ is acyclic. Assume for contradiction that the graph contains a directed cycle C. Choose any node u of this cycle and consider the label $\ell_u(\theta)$. Since $T_e(\theta) \geq \theta + \tau_e$, the labels along the cycles increase at least by the transit times. Since the sum of transit times in all directed cycles is greater than zero by assumption, we have $\ell_u(\theta) \geq \ell_u(\theta) + \sum_{e\in C} \tau_e > \ell_u(\theta)$, which is a contradiction.
By definition the indegree of every node, except for s, is at least one. Since the graph is acyclic it follows, that every node is reachable from s by starting in the respective node and going back along the entering arcs until reaching s. □

A.7 Characterizations of Nash Flows Over Time

Lemma 14. *Let f be a feasible flow over time. We define $\Theta_e := \{ \theta \in [0,\infty) \mid e \in E'_\theta \}$ to be the set of particle starting times for which e is active and $\Theta_e^c := [0,\infty)\backslash\Theta_e$ its complement. The following statements are equivalent.*

(i) f is a Nash flow over time.

(ii) For each arc $e = uv$, it holds that $f_e^+(\theta) = 0$ for almost all $\theta \in \ell_u(\Theta_e^c)$.

(iii) $F_e^+(\ell_u(\theta)) = F_e^-(\ell_v(\theta))$ for all arcs $e = uv$ and all times θ.

(iv) For every arc $e = uv$ and almost all $\theta \in \Theta_e^c$ we have $f_e^+(\ell_u(\theta)) \cdot \ell'_u(\theta) = 0$.

(v) For all θ and every arc $e = uv$ we have:

$$F_e^+(\ell_u(\theta) - \varepsilon) < F_e^+(\ell_u(\theta)) \text{ for all } \varepsilon > 0 \quad \Rightarrow \quad e \in E'_\theta.$$

Proof. *(i)* ⇔ *(ii)*: Note that the flow conservation implies $f_e^+(\theta) = 0$ for all $\theta \in [0, \ell_u(0)]$ for an arc $e = uv$, since flow only starts at time 0 at the source and it cannot reach u faster than $\ell_u(0)$. Furthermore, we have $\ell_u(\theta) \geq \theta$, and therefore ℓ_u is unbounded, or in other words, surjective on $[\ell_u(0), \infty)$.

The contraposition of the Nash flow condition reads $f_e^+(\theta) = 0$ for almost all $\theta \in \ell_u(\Theta_e)^c$. So it is sufficient to show that for almost all $\xi \in [\ell_u(0), \infty)$ we have

$$\xi \in \ell_u(\Theta_e)^c \quad \Leftrightarrow \quad \xi \in \ell_u(\Theta_e^c).$$

"⇒": Let $\theta \geq 0$ with $\ell_u(\theta) = \xi \in \ell_u(\Theta_e)^c$. It follows that $\theta \notin \Theta_e$ and, hence, $\ell_u(\theta) \in \ell_u(\Theta_e^c)$.

"⇐": Let $\xi \in \ell_u(\Theta_e^c)$ and suppose $\xi \notin \ell_u(\Theta_e)^c$, i.e., there are two different times $\theta_1 \in \Theta_e^c$ and $\theta_2 \in \Theta_e$ with $\ell_u(\theta_1) = \xi = \ell_u(\theta_2)$. Since ℓ_u is monotonically increasing, ℓ_u has to be constant between θ_1 and θ_2, and therefore there exists a rational number $\kappa_\xi \in \mathbb{Q}$ with $\ell_u(\kappa_\xi) = \theta$. Since for every point in time $\xi \in \ell_u(\Theta_e^c), \xi \notin \ell_u(\Theta_e)^c$ there is a $\kappa_\xi \in \mathbb{Q}$, the set $\ell_u(\Theta_e^c)\backslash\ell_u(\Theta_e)^c$ is a subset of $\ell_u(\mathbb{Q})$ and, hence, it is countable, and therefore a null set.

(ii) ⇔ *(iii)*: Fix an arc $e = uv$. For all $\theta > 0$ let $I_\theta := (\theta_0, \theta]$, where $\theta_0 \in [0, \theta]$ is the maximal value with $T_e(\ell_u(\theta_0)) = \ell_v(\theta)$ or zero if no such θ_0 exists. Note that $T_e(\ell_u(\theta_0)) > \ell_v(\theta)$ in this case, since $T_e(\ell_u(\theta)) \geq \ell_v(\theta)$ holds in general and $T_e \circ \ell_u$ is continuous.

We show for $\theta' > 0$ that

$$\theta' \in \bigcup_{\theta > 0} I_\theta \quad \Leftrightarrow \quad \theta' \in \Theta_e^c. \tag{8}$$

On the one hand, we have for all $\theta' \in \Theta_e^c \backslash \{ 0 \}$ that $T_e(\ell_u(\theta')) > \ell_v(\theta')$, and therefore there is a $\theta_0 < \theta'$, which implies $\theta' \in I_{\theta'}$. On the other hand, for all $\theta' \in I_\theta$ we have $\ell_v(\theta') < T_e(\ell_u(\theta'))$, which implies $\theta' \in \Theta_e^c$, because otherwise we had by monotonicity of ℓ_v and $T_e \circ \ell_u$ that

$$T_e(\ell_u(\theta')) \leq \ell_v(\theta') \leq \ell_v(\theta) \leq T_e(\ell_u(\theta_0)) \leq T_e(\ell_u(\theta')),$$

and therefore equality, which would contradict the maximality of θ_0. This finishes the proof of (8). Hence, e is not active for all $\theta' \in (\theta_0, \theta]$. Furthermore, we have $F_e^+(\ell_u(\theta_0)) = F_e^-(T_e(\ell_u(\theta_0)) = F_e^-(\ell_v(\theta))$. Note that for $\theta_0 = 0$ we have $0 \geq F_e^-(\ell_v(\theta)) \geq F_e^+(T_e(\ell_u(\theta_0)) = 0$.

Suppose (ii) is given, which means $f_e^+(\xi) = 0$ for almost all $\xi \in \ell_u(I_\theta) = (\ell_u(\theta_0), \ell_u(\theta)]$. This yields

$$F_e^+(\ell_u(\theta)) - F_e^-(\ell_v(\theta)) = F_e^+(\ell_u(\theta)) - F_e^+(\ell_u(\theta_0)) = \int_{\ell_u(\theta_0)}^{\ell_u(\theta)} f_e^+(\xi)\,\mathrm{d}\xi = 0,$$

which shows (iii).

Conversely, suppose that (iii) holds. We have that Θ_e^c is a union of countably many intervals I_θ for which we have

$$\int_{I_\theta} f_e^+(\xi)\,\mathrm{d}\xi = F_e^+(\ell_u(\theta)) - F_e^+(\ell_u(\theta_0)) = F_e^+(\ell_u(\theta)) - F_e^-(\ell_v(\theta)) = 0,$$

which proves (ii).

(ii) $\Leftrightarrow$ *(iv)*: For every arc $e = uv$ the rule of integration by substitution yields

$$\int_{\ell_u(\Theta_e^c)} f_e^+(\xi)\,\mathrm{d}\xi = \int_{\Theta_e^c} f_e^+(\ell_u(\xi)) \cdot \ell_u'(\xi)\,\mathrm{d}\xi.$$

So this either equals zero or not, which shows that (ii) is equivalent to $f_e^+(\ell_u(\theta)) \cdot \ell_u'(\theta) = 0$ for almost all $\theta \in \Theta_e^c$, i.e., equivalent to (iv).

(i) $\Rightarrow$ *(v)*: Suppose we have $F_e^+(\ell_u(\theta) - \varepsilon) < F_e^+(\ell_u(\theta))$ for all $\varepsilon > 0$. Since $F_e^+(\ell_u(\theta)) > 0$ the Nash flow condition implies that e was part of the current shortest paths network at some point in time before θ. Let $\theta' \leq \theta$ be the last point in time with $e \in E'_{\theta'}$. Since e was not in the current shortest paths network in-between θ' and θ there is no inflow during $[\ell_u(\theta'), \ell_u(\theta)]$, i.e., $F_e^+(\ell_u(\theta)) - F_e^+(\ell_u(\theta')) = 0$. This implies by the assumption that $\ell_u(\theta') = \ell_u(\theta)$, and therefore by (1) and the monotonicity of ℓ_v we have

$$\ell_v(\theta) \leq T_e(\ell_u(\theta)) = T_e(\ell_u(\theta')) = \ell_v(\theta') \leq \ell_v(\theta).$$

Thus, we have equality, which means that $e \in E'_\theta$.

(v) $\Rightarrow$ *(iii)*: For $e \in E'_\theta$ we have by Lemma 10 (i) that $F_e^+(\ell_u(\theta)) = F_e^-(T_e(\ell_u(\theta))) = F_e^-(\ell_v(\theta))$. For $e \notin E'_\theta$, let $\theta_0 \in [0, \theta)$ be minimal with $F_e^+(\ell_u(\theta_0)) = F_e^+(\ell_u(\theta))$, which exists due to the contraposition of (v). If $\theta_0 > 0$ then due to minimality $F_e^+(\ell_u(\theta_0) - \varepsilon) < F_e^+(\ell_u(\theta_0))$ for all $\varepsilon > 0$, and therefore by (v) e is active for θ_0. It follows from the observation above, from the monotonicity of F_e^- and ℓ_v, as well as, from Lemma 10 (i) that

$$F_e^+(\ell_u(\theta)) = F_e^+(\ell_u(\theta_0)) = F_e^-(\ell_v(\theta_0)) \leq F_e^-(\ell_v(\theta)) \leq F_e^-(T_e(\ell_u(\theta))) = F_e^+(\ell_u(\theta)).$$

For $\theta_0 = 0$ we have

$$0 \leq F_e^-(\ell_v(\theta)) \leq F_e^-(T_e(\ell_u(\theta))) = F_e^+(\ell_u(\theta)) = F_e^+(\ell_u(\theta_0)) = 0.$$

In both cases we have $F_e^+(\ell_u(\theta)) = F_e^-(\ell_v(\theta))$. □

A.8 Arc Set Relations

Lemma 15. *Given a Nash flow over time the following holds for all times θ:*

(i) $E^*_\theta \subseteq E'_\theta$.

(ii) $\bar{E}_\theta \subseteq E'_\theta$.

(iii) $\ell_u(\theta) < \ell_v(\theta)$ *for all* $e = uv \in \bar{E}_\theta$.

(iv) $E'_\theta = \{\, e = uv \mid \ell_v(\theta) \geq \ell_u(\theta) + \tau_e \,\}$.

(v) $E^*_\theta = \{\, e = uv \mid \ell_v(\theta) > \ell_u(\theta) + \tau_e \,\}$.

Proof. By Lemma 10 (ii) we have $e \in E^*_\theta \;\Leftrightarrow\; z_e(\ell_u(\theta) + \tau_e) > 0$.

(i) Assume we are given an arc $e \in E^*_\theta$. We show that this arc is also active for the respective time, i.e., $e \in E'_\theta$. Either $F^+_e(\ell_u(\theta) - \varepsilon) < F^+_e(\ell_u(\theta))$ for all $\varepsilon > 0$, then $e \in E'_\theta$ by Lemma 14 (v), or there is a $\theta' < \theta$, such that $F^+_e(\ell_u(\theta')) = F^+_e(\ell_u(\theta))$ and $z_e(\ell_u(\theta') + \tau_e) > 0$ by continuity of z. Using Lemma 10 (iii), Lemma 12, and the monotonicity of ℓ_v we get

$$\ell_v(\theta) \leq T_e(\ell_u(\theta)) = T_e(\ell_u(\theta')) = \ell_v(\theta') \leq \ell_v(\theta).$$

Thus, we have equality, which means that $e \in E'_\theta$.

(ii) Next, we show that an arc e, which is full at time $\ell_u(\theta)$, is also active, i.e., $e \in \bar{E}_\theta$ implies $e \in E'_\theta$. Since e is full at time $\ell_u(\theta)$ we have

$$z_e(\ell_u(\theta)) = d_e(\ell_u(\theta)) + F^+_e(\ell_u(\theta) - \tau_e) - F^+_e(\ell_u(\theta)) \geq \sigma_e - \nu^+_e \cdot \tau_e > 0.$$

Therefore, by continuity of z_e and Lemma 2, we have that $f^-_e(\xi) > 0$ for all $\xi \in [\ell_u(\theta) - \delta, \ell_u(\theta)]$ for a small $\delta > 0$. It follows that for all $\varepsilon > 0$ we have $F^-_e(\ell_u(\theta)) - F^-_e(\ell_u(\theta) - \varepsilon) > 0$.

This together with the storage capacity in Lemma 1 (iii) yields

$$\begin{aligned} F^+_e(\ell_u(\theta)) - F^+_e(\ell_u(\theta) - \varepsilon) &= d_e(\ell_u(\theta)) + F^-_e(\ell_u(\theta)) - d_e(\ell_u(\theta) - \varepsilon) - F^-_e(\ell_u(\theta) - \varepsilon) \\ &> \sigma - d_e(\ell_u(\theta) - \varepsilon) \geq 0. \end{aligned}$$

Hence, Lemma 14 (v) implies $e \in E'_\theta$.

(iii) Since e is active due to (ii) this follows directly if $\tau_e > 0$. For $\tau_e = 0$ this follows since $0 < z_e(\ell_u(\phi)) = z_e(\ell_u(\phi) + \tau_e)$ by Lemma 8, and therefore $q_e(\ell_u(\phi)) > 0$ by Lemma 10 (ii). In both cases we have $\ell_u(\phi) < \ell_u(\phi) + \tau_e + q_e(\ell_u(\phi)) = \ell_v(\phi)$.

(iv) From $e \in E'_\theta$ it follows that $\ell_v(\theta) = \ell_u(\theta) + \tau_e + q_e(\ell_u(\theta)) \geq \ell_u(\theta) + \tau_e$. The reverse inclusion follows, since $q_e(\ell_u(\theta)) > 0$ implies $e \in E^*_\theta$ and, by (i), we get $e \in E'_\theta$. For $q_e(\ell_u(\theta)) = 0$ we have

$$\ell_v(\theta) \leq T_e(\ell_u(\theta)) = \ell_u(\theta) + \tau_e \leq \ell_v(\theta),$$

and therefore equality, which shows that e is active for θ.

(v) From $e \in E^*_\theta$ it follows by (i) that e is active, and therefore $\ell_v(\theta) = \ell_u(\theta) + \tau_e + q_e(\ell_u(\theta)) > \ell_u(\theta) + \tau_e$. The reverse inclusion follows, since

$$\ell_u(\theta) + \tau_e < \ell_v(\theta) \leq T_e(\ell_u(\theta)) = \ell_u(\theta) + \tau_e + q_e(\ell_u(\theta)),$$

and thus, necessarily $q_e(\ell_u(\theta)) > 0$, which implies $e \in E^*_\theta$.

□

A.9 Existence of a Feasible α

Lemma 16. *For a given restricted Nash flow over time on $[0, \phi)$ there exists a feasible $\alpha > 0$.*

Proof. By Lemma 15 (iv) and (v) we have $\ell_v(\phi) - \ell_u(\phi) > \tau_e$ for $e = uv \in E^*_\phi$ and $\ell_v(\phi) - \ell_u(\phi) < \tau_e$ for $e = uv \in E \backslash E'_\phi$ and since $F^+_e(\ell_u(\phi)) - F^-_e(\ell_u(\phi)) = d_e(\ell_u(\phi)) < \sigma_e$ for $e = uv \in E'_\phi \backslash \bar{E}_\phi$ we can find an $\alpha_1 > 0$ that satisfies Equations (2), (3) and (5). Since f^-_e is piecewise-constant and right-continuous so is b^+_e, and therefore we can find an $\alpha_2 > 0$ that satisfies Equation (4). Clearly, $\alpha := \min\{\alpha_1, \alpha_2\} > 0$ is feasible. □

A.10 Spillback Model is a Generalization of the Original Model

Remark 17. *The spillback model is a generalization of the deterministic queuing model. If we disable the inflow and the storage capacity, our construction results in the same Nash flow over time in both models.*

Assume we are given an instance of the deterministic queuing model, which is a network $G = (V, E)$ with source s, sink t and arcs equipped with transit times τ_e and outflow capacities ν_e^-. Then, for the spillback model we keep the network and choose, additionally, storage capacities $\sigma_e = \infty$ and inflow capacities bigger than the total outflow capacity of the preceding arcs $\sum_{e \in \delta^-(u)} \nu_e^-$. This ensures that spillback never occurs: Assume for contradiction that there is a node v and a time θ such that the spillback factor $c_v(\theta)$ is strictly smaller than 1. The maximality of $c_v(\theta)$ and the fair allocation condition imply that there is an arc $e = uv$ with $f_e^-(\theta) < b_e^-(\theta)$, i.e., e is throttled. Due to the no slack condition there has to be an outgoing arc $e' = vw$ with $f_{e'}^+(\theta) = b_{e'}^+(\theta)$. This is a contradiction, because e' can never be full and the inflow capacity is always greater than the inflow rate. Considering the spillback thin flow conditions with c_v substituted by 1 shows that (TF5) can be omitted and (TF4) is irrelevant for b_e^+ large enough. Hence, a spillback thin flow matches a thin flow with resetting as it is stated in [2]. Furthermore, we obtain the same bounds on α since $\sigma_e = \infty$, and therefore the conditions (4) and (5) never apply. This shows that a Nash flow over time with spillback in this network equals a Nash flow over time in the original model, and therefore the spillback model is, indeed, a generalization of the deterministic queuing model.

B Remaining Proofs

B.1 Proof of Lemma 1

Lemma 1. *A feasible flow over time satisfies the following conditions for all θ and every arc e:*

(i) Outflow capacity condition: $f_e^-(\theta) \le \nu_e^-$.

(ii) Non-deficit condition: $z_e(\theta) \ge 0$.

(iii) Storage condition: $d_e(\theta) \le \sigma_e$.

Proof. (i) The outflow capacity condition follows immediately from the fair allocation condition.

(ii) The function $z_e(\theta) = F_e^+(\theta - \tau_e) - F_e^-(\theta)$ is continuous for $\theta \in [\tau_e, \infty)$. Assume for contradiction that $z_e(\theta) < 0$ at some point. Let $\theta_0 := \inf \{ \theta \mid z_e(\theta) < 0 \}$ and $\theta_1 > \theta_0$ such that $z_e(\theta) < 0$ for all $\theta \in (\theta_0, \theta_1]$. By continuity we get $z_e(\theta_0) = 0$ and from the fair allocation condition it follows that $f_e^-(\theta) \le b_e^-(\theta) \le f_e^+(\theta - \tau_e)$ for all $\theta \in [\theta_0, \theta_1]$. But this is a contradiction, since

$$0 > z_e(\theta_1) - z_e(\theta_0) = \int_{\theta_0}^{\theta_1} f_e^+(\xi - \tau_e) - f_e^-(\xi) \, \mathrm{d}\xi \ge 0.$$

(iii) The function $d_e(\theta) = F_e^+(\theta) - F_e^-(\theta)$ is continuous for all $\theta \in [0, \infty)$. Again, assume for contradiction that $d_e(\theta) > \sigma_e$ at some point. Let $\theta_0 := \inf \{ \theta \mid d_e(\theta) > \sigma_e \}$ and $\theta_1 > \theta_0$ such that $d_e(\theta) > \sigma_e$ for all $\theta \in (\theta_0, \theta_1]$. By continuity we get $d_e(\theta_0) = \sigma_e$ and from the inflow condition it follows that $f_e^+(\theta) \le f_e^-(\theta)$ for all $\theta \in [\theta_0, \theta_1]$. Again this is a contradiction, since

$$0 < d_e(\theta_1) - d_e(\theta_0) = \int_{\theta_0}^{\theta_1} f_e^+(\xi) - f_e^-(\xi) \, \mathrm{d}\xi \le 0.$$

□

B.2 Proof of Lemma 2

Lemma 2. *For a given network there is an* $\varepsilon > 0$ *such that for every arc* e *with* $z_e(\theta) > 0$ *we have* $f_e^-(\theta) \geq \varepsilon$*, and therefore the waiting time function* q_e *is well-defined.*

Proof. Let $\nu_{\min} \coloneqq \min(\{\, \nu_e^+, \nu_e^- \mid e \in E \,\} \cup \{\, 1 \,\})$ be a lower bound on all in- or outflow capacity of the network and let $\Sigma \coloneqq \max\{\, \sum_{e \in E} \nu_e^+, 1 \,\}$ be an upper bound on the inflow capacities. Furthermore, let $m = |E|$ denote the number of arcs in the network.
We set

$$\varepsilon \coloneqq \left(\frac{\nu_{\min}}{\Sigma}\right)^m \cdot \nu_{\min}.$$

Note that if e is not throttled, we have $f_e^-(\theta) = b_e^-(\theta) = \nu_e^- \geq \nu_{\min} \geq \varepsilon$.
Hence, we consider the case that $e = (u, v)$ is throttled. By repeatedly applying the no slack condition we can construct a sequence of full arcs $(e_1 = (v, v_1), \ldots, e_k = (v_{k-1}, v_k))$, such that all arcs of the sequence except e_k are throttled. This sequence is finite, since the graph of full arcs is acyclic by the no deadlock condition.
For arc e_k it holds that $f_{e_k}^-(\theta) = b_{e_k}^-(\theta) = \nu_{e_k}^-$, since $z_{e_k}(\theta) > 0$ by Lemma 8 and e_k is not throttled by construction.
Next, we determine a lower bound for $f_{e_i}^-(\theta)$ in dependency of $f_{e_{i+1}}^-(\theta)$ for $i = 1, \ldots, k-1$. The fair allocation condition implies $f_{e_i}^-(\theta) = c_{v_i}(\theta) \cdot \nu_{e_i}^-$ since e_i is throttled by construction.
Furthermore, we have

$$\Sigma \cdot c_{v_i}(\theta) \geq \sum_{e' \in \delta^-(v_i)} \nu_{e'}^- \cdot c_{v_i}(\theta) \geq \sum_{e' \in \delta^-(v_i)} f_{e'}^-(\theta) = \sum_{e' \in \delta^+(v_i)} f_{e'}^+(\theta) \geq f_{e_{i+1}}^+(\theta).$$

Putting this together, we get:

$$f_{e_i}^-(\theta) = c_{v_i}(\theta) \cdot \nu_{e_i}^- \geq \frac{f_{e_{i+1}}^+(\theta)}{\Sigma} \cdot \nu_{\min}$$

and recursively applying this inequality, together with $f_{e_i}^-(\theta) = f_{e_i}^+(\theta)$, yields

$$f_e^-(\theta) \geq \prod_{i=1}^{k-1} \left(\frac{\nu_{\min}}{\Sigma}\right) \cdot f_{e_k}^-(\theta) \geq \left(\frac{\nu_{\min}}{\Sigma}\right)^m \cdot \nu_{\min} = \varepsilon.$$

In the second part of the proof we show that

$$\left\{\, q \geq 0 \;\middle|\; \int_{\theta+\tau_e}^{\theta+\tau_e+q} f_e^-(\xi)\ \mathrm{d}\xi = z_e(\theta+\tau_e) \,\right\}$$

is non-empty.
If $z_e(\theta+\tau_e) = 0$ obviously $q_e(\theta) = 0$. So assume $z_e(\theta+\tau_e) > 0$. We distinguish two cases:
In Case 1 there is a $\theta' \geq \theta$ such that $z_e(\theta'+\tau_e) = 0$. This yields

$$0 = z_e(\theta'+\tau_e) = F_e^+(\theta') - F_e^-(\theta'+\tau_e) \geq F_e^+(\theta) - F_e^-(\theta'+\tau_e) = z_e(\theta+\tau_e) - \int_{\theta+\tau_e}^{\theta'+\tau_e} f_e^-(\xi)\ \mathrm{d}\xi.$$

Thus, we can follow that there is a $q \in [0, \theta' - \theta]$ fulfilling the condition.
In Case 2 suppose $z_e(\theta'+\tau_e) > 0$ and, thus, $f_e^-(\theta'+\tau_e) \geq \varepsilon$ for all $\theta' \geq \theta$.
This shows that

$$\int_{\theta+\tau_e}^{\theta+\tau_e+q} f_e^-(\xi)\ \mathrm{d}\xi \geq \int_{\theta+\tau_e}^{\theta+\tau_e+q} \varepsilon\ \mathrm{d}\xi$$

is unbounded for $q \to \infty$ and by the intermediate value theorem there exists q_0 that fulfills the condition.
Hence, the set is non-empty and closed due to the continuity in q of the left hand side, which shows that $q(\theta)$ is well defined. □

B.3 Proof of Theorem 4

Theorem 4 (Derivatives of underlying static flows are spillback thin flows)**.** *The derivatives $x_e'(\theta)$ and $\ell_v'(\theta)$ of a Nash flow over time together with the spillback factors $c_v(\ell_v(\theta))$ form a spillback thin flow for almost all $\theta \in [0, \infty)$ on the current shortest paths network $G_\theta' = (V, E_\theta')$ with resetting on the arcs with queue E_θ^* and inflow bounds $b_e^+(\ell_u(\theta))$.*

Before we prove the theorem we prove the following lemma, which will be of use during the proof of the theorem.

Lemma 18. *For every element e of a finite set E let $g_e\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function that is differentiable almost everywhere and let $g(\theta) \coloneqq \min_{e\in E} g_e(\theta)$ for all $\theta \geq 0$. It holds that*

$$g'(\theta) = \min_{e\in E_\theta'} g_e'(\theta) \tag{9}$$

for almost all $\theta \geq 0$ where $E_\theta' \coloneqq \{\, e \in E \mid g(\theta) = g_e(\theta)\,\}$.

Proof. Let $\theta \geq 0$ such that g as well as all g_e, $e \in E$, are differentiable, which is almost everywhere. Since all functions g_e are continuous at θ we have for sufficiently small $\varepsilon > 0$ that $g(\theta+\xi) = \min_{e\in E_\theta'} g_e(\theta+\xi)$ for all $\xi \in [\theta, \theta+\varepsilon]$. It follows that

$$\begin{aligned} g'(\theta) &= \lim_{\xi\downarrow 0} \frac{g(\theta+\xi) - g(\theta)}{\xi} \\ &= \lim_{\xi\downarrow 0} \min_{e\in E_\theta'} \frac{g_e(\theta+\xi) - g(\theta)}{\xi} \\ &= \min_{e\in E_\theta'} \lim_{\xi\downarrow 0} \frac{g_e(\theta+\xi) - g_e(\theta)}{\xi} \\ &= \min_{e\in E_\theta'} g_e'(\theta). \end{aligned}$$

■

Now we are prepared to prove Theorem 4.

Proof. Let θ be a point in time such that for all $e = uv$ the derivatives of x_e, ℓ_v, and $T_e \circ \ell_u$ exist and $x_e'(\theta) = f_e^-(\ell_v(\theta)) \cdot \ell_v'(\theta) = f_e^+(\ell_u(\theta)) \cdot \ell_u'(\theta)$, which is almost everywhere.
For the sake of clarity, let $\ell_v' \coloneqq \ell_v'(\theta)$, $x_e' \coloneqq x_e'(\theta)$, $c_v \coloneqq c_v(\ell_v(\theta))$, $b_e^+ \coloneqq b_e^+(\ell_u(\theta))$, $E' \coloneqq E_\theta'$, and $E^* \coloneqq E_\theta^*$.

(TF1) We have $\ell_s(\theta) = \theta$ yielding $\ell_s' = 1$. Note that $\delta^-(s) = \emptyset$, and therefore the flow rate leaving s equals r. Hence, for all $e \in \delta^+(s)$ we have by the assumptions of $\sigma_e = \infty$ and $\nu_e^+ > r$ that e is never full, and therefore $b_e^+ = \nu_e^+ > r \geq f_e^+(\ell_s(\theta))$. Hence, the no slack condition implies that no incoming arc is throttled, which is equivalent to $c_s = 1$.

(TF2) Applying the differentiation rule for a minimum (Lemma 18) on

$$\ell_v(\theta) = \min_{e=uv\in E} T_e(\ell_u(\theta))$$

we obtain

$$\ell'_v = \min_{e=uv\in E'} T'_e(\ell_u(\theta)) \cdot \ell'_u.$$

Note that the arcs in E' are exactly the ones with $\ell_v(\theta) = T_e(\ell_u(\theta))$, and therefore the only ones that need to be considered for the derivative.
In the following we analyze the derivative of $T_e(\theta) = \theta + \tau_e + q_e(\theta)$ at the point $\ell_u(\theta)$ for active arcs $e = uv \in E'$. Lemma 11 yields

$$T'_e(\ell_u(\theta)) = \begin{cases} \frac{f_e^+(\ell_u(\theta))}{f_e^-(\ell_v(\theta))} & \text{if } f_e^-(\ell_v(\theta)) > 0 \\ 0 & \text{else if } z_e(\ell_u(\theta)+\tau_e) > 0 \\ 1 & \text{else.} \end{cases}$$

First, we consider the case $f_e^-(\ell_v(\theta)) = 0$, which implies $x'_e = 0$ and hence,

$$T'_e(\ell_u(\theta)) \cdot \ell'_u = \left\{ \begin{array}{ll} 0 & \text{if } q_e(\ell_u(\theta)) > 0 \\ \ell'_u & \text{else} \end{array} \right\} = \rho_e(\ell'_u, x'_e, c_v).$$

Next, we consider the case $f_e^-(\ell_v(\theta)) > 0$ and $x'_e = 0$. In the case of $e \notin E^*$, we have $f_e^+(\ell_u(\theta)) = f_e^+(\ell_v(\theta) - \tau_e) \geq b_e^-(\ell_u(\theta)) \geq f_e^-(\ell_v(\theta)) > 0$, which implies $\ell'_u = x'_e / f_e^+(\ell_u(\theta)) = 0$. This leads to

$$T'_e(\ell_u(\theta)) \cdot \ell'_u = \frac{x'_e}{f_e^-(\ell_v(\theta))} = 0 = \rho_e(\ell'_u, x'_e, c_v).$$

Finally, we consider $f_e^-(\ell_v(\theta)) > 0$ and $x'_e > 0$. This implies that $x_e(\theta) = F_e^+(\ell_u(\theta))$ is increasing. Hence,

$$F_e^+(\ell_u(\theta) + q_e(\ell_u(\theta))) - F_e^+(\ell_u(\theta)) > 0 \quad \Leftrightarrow \quad q_e(\ell_u(\theta)) > 0 \quad \Leftrightarrow \quad e \in E^*.$$

Together with Lemma 10 (v) we obtain

$$b_e^-(\ell_v(\theta)) = \begin{cases} \nu_e^- & \text{if } e \in E^* \\ \min\{\, f_e^+(\ell_u(\theta)), \nu_e^- \,\} & \text{if } e \in E' \backslash E^*. \end{cases}$$

Hence,

$$\begin{aligned} T'_e(\ell_u(\theta)) \cdot \ell'_u &= \frac{x'_e}{f_e^-(\ell_v(\theta))} \\ &= \frac{x'_e}{\min\{\, c_v \cdot \nu_e^-, b_e^-(\ell_v(\theta)) \,\}} \\ &= \begin{cases} x'_e/(c_v \cdot \nu_e^-) & \text{if } e \in E^* \\ \max\left\{ \frac{x'_e}{f_e^+(\ell_u(\theta))}, x'_e/(c_v \cdot \nu_e^-) \right\} & \text{if } e \in E' \backslash E^* \end{cases} \\ &= \rho_e(\ell'_u, x'_e, c_v). \end{aligned} \tag{10}$$

In summary, we have

$$\ell'_v = \min_{e=uv\in E'} T'_e(\ell_u(\theta)) \cdot \ell'_u = \min_{e=uv\in E'} \rho_e(\ell'_u, x'_e, c_v).$$

(TF3) Suppose $x'_e = f_e^-(\ell_v(\theta)) \cdot \ell'_v > 0$. With (10) we get $\ell'_v = x'_e / f_e^-(\ell_v(\theta)) = \rho_e(\ell'_u, x'_e, c_v)$.

(TF4) For every arc $e = vw \in E'$ we know from the inflow condition that $f_e^+(\ell_v(\theta)) \leq b_e^+$. Hence,

$$\frac{x'_e}{b_e^+} = \frac{f_e^+(\ell_v(\theta)) \cdot \ell'_v}{b_e^+} \leq \ell'_v.$$

This shows (TF4).

(TF5) Suppose $c_v < 1$, which means that at least one incoming arc is throttled, since otherwise $c = 1$ would fulfill the fair allocation condition contradicting the maximality of c_v.
By the no slack condition there has to be a full arc $e = vw$ with $f_e^+(\ell_v(\theta)) = b_e^+$ and for this arc we have

$$\frac{x'_e}{b_e^+} = \frac{f_e^+(\ell_v(\theta)) \cdot \ell'_v}{f_e^+(\ell_v(\theta))} = \ell'_v.$$

This, together with (TF4), shows (TF5). □

B.4 Proof of Theorem 5

Theorem 5 (Existence of spillback thin flows)**.** *Consider an acyclic network $G' = (V, E')$ with source s and sink t, such that each node is reachable from s. Furthermore, let $(\nu_e^-)_{e \in E'}$ be outflow capacities, $(b_e^+)_{e \in E'}$ be inflow bounds, and $E^* \subseteq E'$ be a set arcs. Then there exists a spillback thin flow (x', ℓ', c) with resetting on E^*.*

Proof. We prove the existence of a spillback thin flow by converting Equations (TF1) to (TF5) to a continuous function Γ with a compact and convex domain K and consider a variational inequality that asks for a solution $x \in K$ such that $(y - x)^t \Gamma(x) \geq 0$ for all $y \in K$. Since K is an n-dimensional cube a variation of Brouwer's fixed-point theorem [9, 11] shows that such a solution exists. In the next step we show that every solution x does not hit the upper boundary of K, and thus fulfills the non-linear complementary problem: $x \cdot \Gamma(x) = 0$ and $\Gamma(x) \geq 0$. With this and with the structure of the spillback thin flow constraints we show that x corresponds to a spillback thin flow.

Variational Inequality. Let I be an index set with n elements. Given $K \subseteq \mathbb{R}^I$ and a mapping $\Gamma \colon K \to \mathbb{R}^I$, the variational inequality $\mathrm{VI}(K, \Gamma)$ problem is to find a vector $x \in K$ such that

$$(y - x)^t \Gamma(x) \geq 0, \quad \forall y \in K. \tag{VI}$$

We denote the set of all solutions with $\mathrm{SOL}(K, \Gamma)$.
From the Brouwer's fixed-point theorem we can obtain the following [9, 11]:

Theorem 19. *Let $K \subseteq \mathbb{R}^I$ be compact and convex and let $\Gamma \colon K \to \mathbb{R}^I$ be a continuous mapping. Then $\mathrm{SOL}(K, \Gamma)$ is non-empty and compact.*

Nonlinear Complementarity Problem.

Lemma 20. *Let $K := [0, M_1] \times [0, M_2] \times \cdots \times [0, M_n] \subseteq \mathbb{R}^I_{\geq 0}$ be an n-dimensional box, $\Gamma \colon K \to \mathbb{R}^I$ be a continuous mapping, and let $x^* \in \mathrm{SOL}(K, \Gamma)$ be a solution of the variational inequality. For every $i \in I$ with $x_i^* < M_i$ we have*

$$\Gamma_i(x^*) \geq 0 \qquad \text{and} \qquad x_i^* \cdot \Gamma_i(x^*) = 0. \tag{NCP}$$

We call this the complementarity condition *for i.*

Proof. Let $\varepsilon > 0$ with $x_i^* + \varepsilon \leq M_i$. By setting $y := x^* + \varepsilon \cdot e_i \in K$, where e_i is the unit vector of $i \in I$, we obtain from (VI) that $\varepsilon \cdot \Gamma_i(x^*) \geq 0$. Hence $\Gamma_i(x^*) \geq 0$ and since $x_i^* \geq 0$ we also have $x_i^* \cdot \Gamma_i(x^*) \geq 0$. By setting $y' := x^* - x_i^* \cdot e_i \in K$ it follows that $-x_i^* \Gamma_i(x^*) \geq 0$, which shows equality and finishes the proof. ■

To show the existence of a spillback thin flow using this lemma we define a compact, convex set K and a continuous mapping Γ in the following. Let $\bar{V} := \{\, \bar{v} \mid v \in V \,\}$ be a copy of the set of nodes V and consider the index set $I := E' \,\dot\cup\, V \,\dot\cup\, \bar{V}$. Let $e \in E'$ correspond to x'_e, $v \in V$ to ℓ'_v and $\bar{v} \in \bar{V}$ to β_v, which corresponds bijectively to c_v.

Domain K. With $\nu^-_{\min} := \min_{e \in E'} \nu^-_e$, $\nu^-_{\max} := \max_{e \in E'} \nu^-_e$, and $b^+_{\min} = \min_{e \in E'} b^+_e$ we define

$$M := \max \left\{ 1, \frac{r}{\nu^-_{\min}}, \frac{r}{b_{\min}}, \frac{\nu^-_{\max} \cdot |E'|}{b^+_{\min}} \right\}. \tag{11}$$

We define $K \subseteq \mathbb{R}^I_{\geq 0}$ as follows:

$$K := \left\{ (x', \ell', \beta) \in \mathbb{R}^I \;\middle|\; \begin{array}{ll} 0 \leq x'_e \leq 4M^2 \cdot \nu^-_e & \text{for all } e \in E' \\ 0 \leq \ell'_v \leq 3M^2 & \text{for all } v \in V \\ 0 \leq \beta_v \leq \log(2M) & \text{for all } \bar{v} \in \bar{V} \end{array} \right\}$$

Function Γ. We now define the function $\Gamma \colon \mathbb{R}^I_{\geq 0} \to \mathbb{R}^I$ such that we can derive a spillback thin flow from a solution of $\mathrm{VI}(K, \Gamma)$.

$$\Gamma_i(x', \ell', \beta) := \begin{cases} \dfrac{x'_e}{\nu^-_e \cdot e^{-\beta_v}} - \ell'_v & \text{if } i = e = uv \in E^* \\ \max\left\{ \ell'_u, \dfrac{x'_e}{\nu^-_e \cdot e^{-\beta_v}} \right\} - \ell'_v & \text{if } i = e = uv \in E' \setminus E^* \\ \displaystyle\sum_{e \in \delta^-(v)} x'_e - \sum_{e \in \delta^+(v)} x'_e & \text{if } i = v \in V \setminus \{\, s, t \,\} \\ \displaystyle\sum_{e \in \delta^-(t)} x'_e - \sum_{e \in \delta^+(t)} x'_e - r & \text{if } i = t \in V \\ \ell'_s - \dfrac{1}{e^{-\beta_s}} & \text{if } i = s \in V \\ \ell'_v - \displaystyle\max_{e = vw \in E'} \dfrac{x'_e}{b^+_e} & \text{if } i = \bar{v} \in \bar{V} \end{cases}$$

Clearly, K is convex and compact and Γ is continuous, and therefore by Theorem 19 we have a solution $(x', \ell', \beta) \in \mathrm{SOL}(K, \Gamma)$.

Lemma 21. *For every solution $(x', \ell', \beta) \in \mathrm{SOL}(K, \Gamma)$ we have*

(i) $x'_e < 4M^2 \cdot \nu^-_e$ *for every arc* e,

(ii) $\ell'_v < 3M^2$ *for every node* $v \in V$,

(iii) $\beta_v < \log(2M)$ *for every node* $v \in V \setminus \{\, s \,\}$ *with* $\sum_{e \in \delta^+(v)} x'_e > 0$.

Proof. We prove this lemma in three steps.

(i) Suppose there is an arc $e \in E'$ with $x'_e = 4M^2 \cdot \nu_e^-$. For $(y, \ell', \beta) \in K$ with $y_e := 0$, $y_i := x'_i$ for $i \in E' \setminus \{ e \}$, we have

$$0 \leq -x'_e \cdot \Gamma(x', \ell', c) \leq x'_e \cdot \left(\ell'_v - \frac{x'_e}{\nu_e^- \cdot e^{-\beta_v}} \right) \leq 4M^2 \cdot \nu_e^- \cdot (\ell'_v - 4M^2).$$

Note that $e^{-\beta_v} \leq 1$. In other words, $\ell'_v - 4M^2 \geq 0$ which is a contradiction since $\ell'_v \leq 3M^2$. Therefore, we have for every $e \in E'$ that $x'_e < 4M^2 \cdot \nu_e^-$ and by Lemma 20 we obtain the complementarity condition:

$$\Gamma_e(x', \ell', \beta) \geq 0 \qquad \text{and} \qquad x'_e \cdot \Gamma_e(x', \ell', \beta) = 0. \tag{12}$$

(ii) Using (x', k, β) with $k_v = \ell'_v$ for $v \neq s$ we obtain with (VI) that $(k_s - \ell'_s) \cdot (\ell'_s - 1/e^{-\beta_s}) \geq 0$ for all $k_s \in [0, 3M^2]$. Hence, $\ell'_s = 1/e^{-\beta_s} \leq 2M < 3M^2$.

For every node $v \in V \setminus \{ s, t \}$ we can show that

$$\sum_{e \in \delta^-(v)} x'_e \leq \sum_{e \in \delta^+(v)} x'_e. \tag{13}$$

If $\ell_v > 0$ this follows from (VI) for $(x, k, \beta) \in K$ with $k_u = \ell_u$ for all nodes $u \in V \setminus \{ v \}$ and $k_v = 0$. For $\ell_v = 0$ this inequality holds because the equation on the right of (12) implies that $x'_e = 0$ on all arcs $e \in \delta^-(v)$.

If we define $b(v) := \sum_{e \in \delta^+(v)} x'_e - \sum_{e \in \delta^-(v)} x'_e$ for all $v \in V$ the flow x'_e is a feasible static b-transshipment, where $b(v) \geq 0$ for all $v \in V \setminus \{ t \}$ (note that s has no incoming arcs). Since the graph G' is acyclic and t is the only sink in this b-transshipment, we get that $\sum_{e \in \delta^+(t)} x'_e = 0$, and therefore the definition of Γ_t together with (VI) implies that $\sum_{e \in \delta^-(t)} x'_e$ is less or equal to r, i.e., $b(t) \geq -r$. To summarize this it can be said that x' is a b-transshipment of value at most r. In the following we show that a label of $3M^2$ would induce a flow of $x'_e > r$ on an arc. Clearly, this would be a contradiction since no arc can carry more flow than the total transshipment value.

Let us now consider a node w with label $3M^2$. Since every node is reachable by s we consider an s-w-path. Since $\ell'_s < 3M^2$, there has to be an arc $a = uv$ on the path, such that $\ell'_u < \ell'_v = 3M^2$. We distinguish two cases. If $x'_a = 0$ choose a vector $(y, \ell', \beta) \in K$ with $y_e := x'_e$ for all arcs $e \neq a$ and $y_a > 0$. By inserting this into (VI), we get

$$0 \leq \begin{cases} y_a \cdot (\ell'_u - \ell'_v) & \text{if} \quad a \in E' \setminus E^* \\ y_a \cdot (0 - \ell'_v) & \text{if} \quad a \in E^*. \end{cases}$$

This leads to a contradiction since $y_a > 0$ and $(\ell'_u - \ell'_v) < 0$ as well as $(0 - \ell'_v) < 0$. We therefore suppose $x'_a > 0$. Since $\ell'_u < \ell'_v$ we get with (12) and the definition of Γ_a that $x'_a = \ell'_v \cdot \nu_a^- \cdot e^{-\beta_v}$ and thus

$$x'_a = \ell'_v \cdot \nu_a^- \cdot e^{-\beta_v} \geq 3M^2 \cdot \nu_{\min}^- \cdot e^{-\log(2M)} = \frac{3M}{2} \cdot \nu_{\min}^- > M \cdot \nu_{\min}^- \overset{(11)}{\geq} r.$$

This is a contradiction as we have shown above. Thus, $\ell'_v < 3M^2$ for every $v \in V$. Lemma 20 and (13) imply that x' is an s-t-flow of value r, i.e.,

$$\sum_{e \in \delta^+(v)} x'_e - \sum_{e \in \delta^-(v)} x'_e = \begin{cases} r & \text{if } v = s \\ -r & \text{if } v = t \\ 0 & \text{else.} \end{cases} \tag{14}$$

(iii) Suppose $\beta_v = \log(2M)$ for some $v \in V$ with $\sum_{e\in\delta^+(v)} x'_e > 0$. For $(x', \ell', \gamma) \in K$ with $\gamma_u := \beta_u$ for all $u \neq v$ and $\gamma_v := 0$ we obtain from (VI) that

$$\ell'_v \leq \max_{e=vw\in E} \frac{x'_e}{b^+_e}. \tag{15}$$

Let $e = vw$ be an arc for which $x'_e/b^+_e > 0$ is maximal. For $v = s$ we have

$$\ell'_s = \frac{1}{e^{-\beta_s}} = 2M > \frac{r}{b^+_{\min}} \geq \frac{x'_e}{b^+_e}.$$

For $v \neq s$ Equation (14) implies that there is at least one incoming arc $e' = uv$ that carries $x'_{e'} \geq x'_e/|\delta^-(v)| \geq x'_e/|E'| > 0$ flow. Using the right side of (12) for e' yields $\Gamma_{e'}(x', \ell', \beta) = 0$, and therefore

$$\ell'_v \geq \frac{x'_{e'}}{\nu^-_{e'} \cdot e^{-\beta_v}} \geq \frac{x'_e \cdot e^{\log(2M)}}{|E'| \cdot \nu^-_{e'}} \overset{(11)}{\geq} \frac{x'_e \cdot 2 \cdot \nu^-_{\max} \cdot |E'|}{|E'| \cdot \nu^-_{e'} \cdot b^+_{\min}} > \frac{x'_e}{b^+_{\min}} \geq \frac{x'_e}{b^+_e}.$$

In both cases we have a contradiction to (15).

■

Constructing the spillback thin flow. Let $(x', \tilde{\ell}', \beta)$ be a solution to VI(K, Γ). In order to obtain a spillback thin flow we need to make some modifications. Let $V_0 \subseteq V \setminus \{ s \}$ be the set of nodes with $\sum_{e\in\delta^-(v)} x'_e = \sum_{e\in\delta^+(v)} x'_e = 0$. We set

$$c_v := \begin{cases} 1 & \text{if } v \in V_0 \\ e^{-\beta_v} & \text{else.} \end{cases}$$

Note that we have $\rho_e(\cdot, x'_e, e^{-\beta_v}) = \rho_e(\cdot, x'_e, c_v)$ because $c_v \neq e^{-\beta_v}$ implies $x'_e = 0$. Furthermore, let

$$L := \left\{ k \in \mathbb{R}^V_{\geq 0} \;\middle|\; k_v = \tilde{\ell}_v \text{ for } v \in V \setminus V_0 \text{ and } k_v \leq \min_{e=uv\in E'} \rho_e(k_u, x'_e, c_v) \text{ for } v \in V \right\}.$$

Clearly, $\tilde{\ell}' \in L$, because for every $v \in V$ we obtain by the left side of Lemma 20 applied to $e = uv$ that

$$\tilde{\ell}_v \leq \left\{ \begin{array}{ll} x'_e/(\nu^-_e \cdot e^{-\beta_v}) & \text{if } e \in E^* \\ \max\{ \tilde{\ell}'_u, x'_e/(\nu^-_e \cdot e^{-\beta_v}) \} & \text{if } e \in E' \setminus E^* \end{array} \right\} = \rho_e(\tilde{\ell}'_u, x'_e, e^{-\beta_v}) = \rho_e(\tilde{\ell}'_u, x'_e, c_v).$$

So L is non-empty and closed. From the facts that x'_e and $\tilde{\ell}'_s = \frac{1}{e^{-\beta_s}} \leq 2M$ are bounded and every node is reachable from s this set is also bounded, i.e., the following is well-defined:

$$\ell' := \arg\max_{k\in L} \sum_{v\in V} k_v.$$

In the remaining of the proof we show that the constructed triple (x', ℓ', c) forms a spillback thin flow. Equation (14) states that x' is a static s-t-flow of value r, so it remains to show that Equations (TF1) to (TF5) are fulfilled, which we do by applying Lemma 20 to the respective variables.

(TF1) From (NCP) applied to ℓ'_s it follows that $(\ell'_s - \frac{1}{e^{-\beta_s}}) \geq 0$ and $\ell'_s \cdot (\ell'_s - \frac{1}{e^{-\beta_s}}) = 0$. Thus, $\ell'_s = 1/e^{-\beta_s}$ and since $e^{-\beta_s} = c_s$ (TF1) is fulfilled.

(TF3) From (NCP) applied to x'_e we get

$$x'_e \cdot \left(\rho_e(\tilde{\ell}'_u, x'_e, e^{-\beta_v}) - \tilde{\ell}'_v\right) = 0.$$

So if $x'_e > 0$ it follows that $u, v \notin V_0$, and therefore $\tilde{\ell}'_v = \ell'_v$ and $\tilde{\ell}'_u = \ell'_u$, which shows $\ell'_v = \rho_e(\ell'_u, x'_e, e^{-\beta_v}) = \rho_e(\ell'_u, x'_e, c_v)$.

(TF2) By the definition of L we obtain $\ell'_v \leq \rho_e(\ell'_u, x'_e, c_v)$. In order to show equality we consider the following two cases. If $v \notin V_0$ there has to be at least one incoming arc $e = uv$ with $x'_e > 0$, and therefore equality follows from (TF3).
For $v \in V_0$ we suppose for contradiction that $\ell'_v < \min_{e=uv \in E'} \rho_e(\ell'_u, x'_e, c_v)$. Let $k_w := \ell'_w$ for $w \in V \setminus \{ v \}$ and $k_v = \min_{e=uv \in E'} \rho_e(\ell'_u, x'_e, c_v)$. Since $\rho_e(\cdot, x'_e, c_w)$ is monotonically increasing we have for $w \neq v$ that $k_w = \ell'_w \leq \rho_e(\ell'_u, x'_e, c_w) \leq \rho_e(k_u, x'_e, c_w)$ and for $w = v$ the condition holds by definition. Hence, $k_v \in L$ which is a contradiction to the maximality of ℓ'_v, since $\sum_{v \in V} k_v > \sum_{v \in V} \ell'_v$.

(TF4) From (NCP) applied to $\bar{v} \in \bar{V}$ we get $\tilde{\ell}'_v - \max x'_e / b^+_e \geq 0$, which proves (TF4) for $v \notin V_0$. For $v \in V_0$ we have $\ell'_v \geq 0 = \max\limits_{e=vw \in E'} \frac{x'_e}{b^+_e}$, trivially.

(TF5) Finally, we have

$$\beta_v \cdot \left(\tilde{\ell}_v - \max_{e=vw \in E'} \frac{x'_e}{b^+_e}\right) = 0$$

which implies (TF5) for $v \notin V_0$, since $\beta_v > 0$ means that $c_v = e^{-\beta_v} < 1$ and thus we have equality in this case. For $v \in V_0$ we set $c_v = 1$, and therefore there is nothing to show.
Hence, (x', ℓ', c) forms a spillback thin flow, which finishes the proof of Theorem 5. □

B.5 Proof of Theorem 6

Theorem 6 (α-Extensions are restricted Nash flows over time)**.** *Given a restricted Nash flow over time on $[0, \phi)$ and a feasible $\alpha > 0$, the α-extension is a restricted Nash flow over time on $[0, \phi + \alpha)$. Furthermore, the extended ℓ-functions are indeed the earliest arrival times and the extended x-functions describe the underlying static flow for all $\theta \in [0, \phi + \alpha)$.*

Proof. Obviously f^-_e and f^+_e are bounded, piece-wise constant, and right-continuous. It is clear that all conditions are fulfilled on $[0, \phi)$ as well as on $[\phi + \alpha, \infty)$ since nothing has changed on this intervals. Note that in the first part of the proof we use the linearly extended ℓ-labels and we show only in the end that they are indeed the earliest arrival times.

Flow conservation. First, we show that the α-extension satisfies the flow conservation. For $\ell'_v > 0$ we obtain

$$\sum_{e \in \delta^+(v)} f^+_e(\theta) - \sum_{e \in \delta^-(v)} f^-_e(\theta) = \sum_{e \in \delta^+(v)} \frac{x'_e(\theta)}{\ell'_v} - \sum_{e \in \delta^-(v)} \frac{x'_e(\theta)}{\ell'_v} = \begin{cases} 0 & \text{if } v \in V \setminus \{ s, t \} \\ r & \text{if } v = s \end{cases}$$

for all $v \in V \setminus \{ t \}$ and all $\theta \in [\ell_v(\phi), \ell_v(\phi + \alpha))$. Here we use that $\ell'_s = \frac{1}{c_s}$ and $c_s = 1$ by assumptions made in Section 3 and as argued in the Proof of Theorem 4. For the case $\ell'_v = 0$ we have $[\ell_v(\phi), \ell_v(\phi + \alpha)) = \emptyset$, so there is nothing to show.

x is well-defined. For all $\xi \in [0, \alpha)$ we have

$$F_e^+(\ell_u(\phi + \xi)) = x_e(\phi) + \int_{\ell_u(\phi)}^{\ell_u(\phi)+\xi\cdot\ell'_u} f_e^+(\theta)\,\mathrm{d}\theta = x_e(\phi) + \xi \cdot x'_e = x_e(\phi + \xi).$$

It follows analogously that $F_e^-(\ell_v(\phi + \xi)) = x_e(\phi + \xi)$, and therefore

$$F_e^+(\ell_u(\phi + \xi)) = F_e^-(\ell_v(\phi + \xi)) = x_e(\phi + \xi). \tag{16}$$

Fair allocation condition. For the fair allocation condition we have to show for every arc $e = uv$ that $f_e^-(\ell_v(\phi + \xi)) = \min\{ b_e^-(\ell_v(\phi + \xi)), c_v \cdot \nu_e^- \}$ for $\xi \in [0, \alpha)$, which states that the outflow rate operates as intended. Since this is obvious for $\ell'_v = 0$, we assume $\ell'_v > 0$ and distinguish three cases:

Case 1 $\quad x'_e = 0$
Either e is not active or it is active but $\ell'_v > 0$ and (TF2) implies that e is not resetting. Either way $z_e(\ell_u(\phi)) = 0$ and since $f_e^+(\ell_u(\phi + \xi)) = 0$ the queue stays empty. For the left hand side we have $f_e^-(\ell_v(\phi + \xi)) = x'_e/\ell'_v = 0$ for $\xi \in [0, \alpha)$. For the right hand side we have $b_e^-(\ell_v(\phi + \xi)) = f_e^+(\ell_v(\phi + \xi) - \tau_e) = 0$ since either $\ell_v(\phi + \xi) - \tau_e \geq \ell_u(\phi)$ (so it is part of the current spillback thin flow or comes afterwards and in either case the inflow is zero), or $\ell_v(\phi + \xi) - \tau_e < \ell_u(\phi)$ (so the inflow comes from earlier than our current spillback thin flow). In the later case e is not active for ζ with $\ell_u(\zeta) = \ell_v(\phi + \xi) - \tau_e$, since $T_e(\zeta) = \ell_u(\zeta) + \tau_e + q_e(\zeta) \geq \ell_v(\phi + \xi) > \ell_v(\zeta)$. We constructed our flow over time in such a way that the Nash flow condition is fulfilled for every point in time, and therefore we have $f_e^+(\ell_v(\phi + \xi) - \tau_e) = f_e^+(\ell_u(\zeta)) = 0$ for all $\xi \in [0, \alpha)$.

Case 2 $\quad x'_e > 0$ and $e \in E'_\phi \backslash E^*_\phi$ with $x'_e/(c_v \cdot \nu_e^-) \leq \ell'_u$
It follows from (TF3) that $\ell'_v = \ell'_u$ and thus $f_e^+(\ell_u(\phi + \xi)) = x'_e/\ell'_u = x'_e/\ell'_v = f_e^-(\ell_v(\phi + \xi))$ for $\xi \in [0, \alpha)$. We obtain

$$f_e^+(\ell_v(\phi+\xi) - \tau_e) = f_e^+(\ell_v(\phi) - \tau_e + \ell'_v \cdot \xi)) = f_e^+(\ell_u(\phi) + \ell'_u \cdot \xi)) = f_e^+(\ell_u(\phi+\xi)) = f_e^-(\ell_v(\phi+\xi)).$$

Using this equality we can show that $z_e(\ell_v(\phi+\xi)) = z_e(\ell_v(\phi)) + \int_{\ell_v(\phi)}^{\ell_v(\phi+\xi)} f_e^+(\xi - \tau_e) - f_e^-(\xi)\,\mathrm{d}\xi = 0$. By the case distinction we have $b_e^-(\ell_v(\phi+\xi)) = f_e^+(\ell_v(\phi+\xi) - \tau_e) = x'_e/\ell'_u \leq c_v \cdot \nu_e^-$. In conclusion we have

$$\min\{ b_e^-(\ell_v(\phi + \xi)), c_v \cdot \nu_e^- \} = b_e^-(\ell_v(\phi + \xi)) = f_e^+(\ell_v(\phi + \xi) - \tau_e) = f_e^-(\ell_v(\phi + \xi)).$$

Case 3 $\quad x'_e > 0$ and ($e \in E^*_\phi$ or $e \in E'_\phi \backslash E^*_\phi$ with $x'_e/(c_v \cdot \nu_e^-) > \ell'_u$)
It follows from (TF3) that $\ell'_v = x'_e/(c_v \cdot \nu_e^-)$ and thus $f_e^-(\ell_v(\phi+\xi)) = x'_e/\ell'_v = c_v \cdot \nu_e^-$ for $\xi \in [0, \alpha)$. It remains to show that $b_e^-(\ell_v(\phi+\xi)) \geq c_v \cdot \nu_e^-$. For $e \in E^*_\phi$ we get from (2) that $\ell_v(\phi) - \ell_u(\phi) + \xi \cdot (\ell'_v - \ell'_u) > \tau_e$ for $\xi \in [0, \alpha)$. For $e \in E'_\phi \backslash E^*_\phi$ and $\ell'_v = x'_e/(c_v \cdot \nu_e^-) > \ell'_u$ it follows that $\ell_v(\phi) - \ell_u(\phi) = \tau_e$ and $\xi \cdot (\ell'_v - \ell'_u) > 0$ for $\xi \in (0, \alpha)$. In both cases we get that $\ell_v(\phi+\xi) - \tau_e > \ell_u(\phi+\xi)$ for $\xi \in (0, \alpha)$. We use this to show that $z_e(\ell_v(\phi + \xi)) > 0$:

$$\begin{aligned} z_e(\ell_v(\phi + \xi)) &= F_e^+(\ell_v(\phi + \xi) - \tau_e) - F_e^-(\ell_v(\phi + \xi)) \\ &\overset{(16)}{=} F_e^+(\ell_v(\phi + \xi) - \tau_e) - F_e^+(\ell_u(\phi + \xi)) \\ &\geq F_e^+(\ell_u(\phi + \xi) + \varepsilon) - F_e^+(\ell_v(\phi + \xi)) = \varepsilon \cdot \frac{x'_e}{\ell'_u} > 0, \end{aligned}$$

where we choose $\varepsilon > 0$, such that $\ell_u(\phi + \xi) + \varepsilon < \min\{ \ell_u(\phi + \alpha), \ell_v(\phi + \xi) - \tau_e \}$. Then the first inequality follows by monotonicity of F_e^+.
Note that since a flow of x'_e leaves node u there either has to be some inflow of x' into u or $u = s$. In both cases we have $\ell'_u > 0$, and thus $\ell_u(\phi + \xi) < \ell_u(\phi + \alpha)$ and x'_e/ℓ'_u is well-defined.

Inflow condition and no slack condition. For all $\xi \in [0, \alpha)$ we show that $f_e^+(\ell_u(\phi + \xi)) \le b_e^+(\ell_u(\phi + \xi))$ holds with equality, whenever there is an incoming throttled arc. Equation (5) ensures that arcs $e \notin \bar{E}_\phi$ stay non-full during $[\ell_u(\phi), \ell_u(\phi + \alpha))$. Together with (4) we get that $b_e^+(\ell_u(\phi + \xi)) = b_e^+$ for all $\xi \in [0, \alpha)$ and hence (TF4) shows that the inflow condition holds for all $\xi \in [0, \alpha)$ since

$$f_e^+(\ell_u(\phi + \xi)) = x'_e / \ell'_u \le b_e^+ = b_e^+(\ell_u(\phi + \xi)).$$

As we have shown in the fair allocation condition an incoming throttled arc implies $c_u < 1$ and thus the inequality holds due to (TF5) with equality. This proves that, the no slack condition is satisfied.

No deadlock condition. Note that full arcs are always active by Lemma 15 (ii), thus they are part of E'_ϕ. Additionally, arcs can only become full arcs if they are flow-carrying, which is only the case for arcs in E'_ϕ. But $G'_\phi = (V, E'_\phi)$ is acyclic, which shows the no deadlock condition.

Earliest arrival times. We show that the extended ℓ-labels fulfill Equation (1), and therefore describe the earliest arrival times. As shown before we have $\ell'_s = 1$ implying $\ell_s(\theta) = \theta$ for all $\theta \in [0, \phi + \alpha)$. Considering $v \ne s$, $e = uv \in E$, and $\xi \in [0, \alpha)$, we distinguish two cases and show $\ell_v(\phi + \xi) \le T_e(\ell_u(\phi + \xi))$ in the first case and $\ell_v(\phi + \xi) = T_e(\ell_u(\phi + \xi))$ in the second case.

Case 1: $e \in E \backslash E'_\phi$ or $e \in E'_\phi \backslash E^*_\phi$ with $\ell'_v < \ell'_u$.
We have for all $\xi \in [0, \alpha)$ that

$$\ell_v(\phi + \xi) = \ell_v(\phi) + \xi \cdot \ell'_v \le \ell_u(\phi) + \tau_e + \xi \cdot \ell'_u \le T_e(\ell_u(\phi) + \xi \cdot \ell'_u) = T_e(\ell_u(\phi + \xi)),$$

where the first inequality follows by (3) for $e \in E \backslash E'_\phi$ or by $\ell_v(\phi) = \ell_u(\phi) + \tau_e$ and $\ell'_v < \ell'_u$ otherwise.

Case 2: $e \in E^*_\phi$ or $e \in E'_\phi \backslash E^*_\phi$ with $\ell'_v \ge \ell'_u$.
If $x'_e = 0$ and $e \in E^*_\phi$ we get from (TF2) that $\ell'_v = \rho_e(\ell'_u, x'_e, c_v) = 0$. Since e is active for ϕ it follows that

$$\ell_v(\phi + \xi) = \ell_v(\phi) = T_e(\ell_u(\phi)) \le T_e(\ell_u(\phi + \xi)). \tag{17}$$

In order to show equality we have

$$q_e(\ell_u(\phi + \xi)) = T_e(\ell_u(\phi + \xi)) - \ell_u(\phi + \xi) - \tau_e \overset{(2)}{>} T_e(\ell_u(\phi + \xi)) - \ell_v(\phi + \xi) \overset{(17)}{\ge} 0.$$

Thus, by Lemma 10 (ii) we have $z_e(\ell_u(\phi + \xi) + \tau_e) > 0$. Lemma 10 (iv) together with $F_e^+(\ell_u(\phi + \xi)) - F_e^+(\ell_u(\phi)) = \xi \cdot x'_e = 0$ implies $T_e(\ell_u(\phi)) = T_e(\ell_u(\phi + \xi))$, and therefore equality in (17).
If we have $x'_e = 0$ and $e \in E'_\phi \backslash E^*_\phi$ with $\ell'_v \ge \ell'_u$ we obtain $\ell'_v \le \rho_e(\ell'_u, x'_e, c_v) = \ell'_u \le \ell'_v$, and therefore $\ell'_v = \ell'_u$. This yields

$$\ell_v(\phi + \xi) = \ell_v(\phi) + \xi \cdot \ell'_v = \ell_u(\phi) + \tau_e + \xi \cdot \ell'_u = \ell_u(\phi + \xi) + \tau_e = T_e(\ell_u(\phi + \xi)),$$

where the last equality holds since there is no inflow within $(\ell_u(\phi), \ell_u(\phi + \xi))$, and therefore no queue.
In the following we assume $x'_e > 0$, which implies $\ell'_v = \rho_e(\ell'_u, x'_e, c_v) > 0$. For all $\xi \in [0, \alpha)$ we have

$$\ell_u(\phi + \xi) + \tau_e = \ell_u(\phi) + \tau_e + \xi \cdot \ell'_u \le \ell_v(\phi) + \xi \cdot \ell'_v = \ell_v(\phi + \xi), \tag{18}$$

where the inequality follows either from (2) in the case of $e \in E^*_\phi$ or, in the other case, by $\ell_v(\phi) = \ell_u(\phi) + \tau_e$ and $\ell'_u \le \ell'_v$. By definition of q_e and z_e we obtain that $q_e(\ell_u(\phi + \xi))$ is the minimal non-negative value with

$$F_e^-(\ell_u(\phi + \xi) + \tau_e + q_e(\ell_u(\phi + \xi))) = F_e^+(\ell_u(\phi + \xi)).$$

Note that F_e^- has the following properties: it is non-decreasing, $F_e^-(\ell_v(\phi+\xi)) = F_e^+(\ell_u(\phi+\xi))$ by (16), and it is strictly increasing at $\ell_v(\phi+\xi)$ with slope $f_e^-(\ell_v(\phi+\xi)) = x'_e/\ell'_v > 0$. With (18) this implies that $q_e(\ell_u(\phi+\xi))$ is uniquely determined and satisfies

$$T_e(\ell_u(\phi+\xi)) = \ell_u(\phi+\xi) + \tau_e + q_e(\ell_u(\phi+\xi)) = \ell_v(\phi+\xi).$$

Both cases together show that for all $v \in V \setminus \{ s \}$ and all $\xi \in [0, \alpha)$ we have

$$\ell_v(\phi+\xi) \leq \min_{e=uv \in E} T_e(\ell_u(\phi+\xi)).$$

It remains to show that equality holds. By (TF2) there has to be an arc $e = uv \in E'$ with $\ell'_v = \rho_e(\ell'_u, x'_e, c_v)$. It follows immediately that if $e \notin E^*$ we have $\ell'_v \geq \ell'_u$. But this means that e belongs to the second case and there we showed that $\ell_v(\phi+\xi) = T_e(\ell_u(\phi+\xi))$.

Nash flow condition. Since all conditions are fulfilled, we have a feasible flow over time. By Lemma 14 (iii) we get from Equation (16) that the Nash flow condition is fulfilled. Note that by construction, the condition holds for every point in time and not only for almost every point in time. □