Dissertation

Nash Flows Over Time

Leon Sering

Nash Flows Over Time

vorgelegt von

M. Sc. **Leon Sering**

ORCID: 0000-0003-2953-1115

an der Fakultät II – Mathematik und Naturwissenschaften

der Technischen Universität Berlin

zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften

– Dr. rer. nat. –

genehmigte Dissertation

Promotionsausschuss:

Vorsitzender	Prof. Dr. John M. Sullivan
Gutachter	Prof. Dr. Martin Skutella
	Prof. Dr. José Correa
	Prof. Dr. Neil Olver

Tag der wissenschaftlichen Aussprache: 28. September 2020

Berlin 2020

Leon Sering
Nash Flows Over Time
Dissertation, Berlin 2020
Reviewers: Prof. Dr. Martin Skutella, Prof. Dr. José Correa and Prof. Dr. Neil Olver

Technische Universität Berlin
Combinatorial Optimization & Graph Algorithms
Institute of Mathematics
Straße des 17. Juni 136
10623 Berlin

Acknowledgments

This thesis is the result of three great years in the *Combinatorial Optimization & Graph Algorithm* (*COGA*) research group at TU Berlin and I met a lot of wonderful people along the way.

First of all, I want to thank my advisor **Martin Skutella**. He hired me, even though I was calling for the job interview from somewhere in the middle of nowhere in Australia with a very rough connection. I am especially thankful that he introduced me to this amazing topic of Nash flows over time and for all the inspiring ideas he contributed during the years. He also made it possible that I had not to worry about any financial barriers and could visit a lot of international conferences and workshops. In this regard, I shall not forget to mention and thank the **Reseach Center Matheon** and **MATH+** for all the funding they provided.

I was very lucky to meet **Laura Vargas Koch** in December 2017. It is no understatement to say that this was a huge boost to my research. It became clear very quickly that we have a very similar approach on mathematical problems and the result of this highly productive collaboration can be seen in this thesis. Thank you very much, for all the great discussions and ideas, for the productive time and the great research visits, and last but not least, for your friendship.

I also want to thank **José Correa** for inviting me to Chile and all the further participants of the two very nice Nash flow workshops in Santiago de Chile and in Berlin: **Neil Olver**, **Laura Vargas Koch**, **Tim Oosterwijk**, **Andrés Cristi**, **Dario Frascaria** and **Marcus Kaiser**. It was a great time in Chile and a pleasure to host you in Berlin. You are truly great people to work with.

I want to thank my co-authors **Lukas Graf** and **Tobias Harks** for the great collaboration and interesting discussions and **Britta Peis** for inviting me to Aachen and providing some great ideas. I am very grateful to **Theresa Thunig** for all of her great insights into the traffic science community and the simulation software *MATSim* and to **Miriam Schlöter** and **Veerle Timmermans** for the great scientific and non-scientific discussions in the last years. I should not forget to thank **Max Zimmer** for his great work on the *Nash Flow Computation Tool*. You did a great implementation job and this tool saved me so many hours of constructing examples.

Finally, I want to thank my girlfriend **Jennifer Manke** for all the great years, the huge support during my doctoral studies, and in particular for spell-checking this thesis. Without your help there would be a lot less pleasure in my life and a lot more typos in the following text. Thank you for everything. I love you.

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1 Introduction

Climate change is inevitable. Right now humanity is facing the biggest crisis since the end of the cold war. Every year we produce over 35 gigatons of carbon dioxide (CO_2) and more than 15 gigatons of other atmospheric greenhouse gases, such as methane (CH_4), nitrous oxide (N_2O) and tropospheric ozone (O_3) [13, 79]. If these anthropogenic emissions persist during the next decades the global temperature will rise between 2.6 °C and 4.8 °C by 2100 causing a rise of the sea level by an estimated 45 to 82 cm [14]. Besides, the *Intergovernmental Panel on Climate Change* (IPCC) reported the possible consequences on our planet in 2014: Extreme weather with storms, floods and droughts will cause water scarcity, crop losses and mass damage to the biodiversity, just to name some examples [12].

Even though the scientific world agrees that anthropogenic activities will inevitably lead to climate change [102], the IPCC recommends to maintain the greenhouse gases below a limit of 450 ppm (parts per million) in order to avoid an increase of more than 2 °C in global temperature [13]. This can only be achieved by starting to reduce the emissions of atmospheric greenhouse gases immediately.

Hereby, the transportation sector is the second biggest primary source of greenhouse gases, right after electricity and heat. This sector alone is responsible for nearly half of the fossil fuel consumptions and an estimated 15% of the global anthropogenic greenhouse emissions [13].

Fortunately, there are several promising technologies and concepts on the horizon that could reduce the emissions significantly. Firstly, and most importantly, changing car engines from combustion fossil fuel to electronic power [15]. This would reduce the emission greatly under the assumption that the energy sector also heavily invests into renewable energies and reduces the usage of fossil energy drastically. Secondly, reducing the individual traffic volume should be one of the primary goals. Even though the global population and with it the transportation demand is growing exponentially, concepts like pooling (sharing a car for a ride with other travelers) [95] or car sharing [88] are promising concepts as the production of cars is very CO_2 costly.

Finally, one of the major goals should be to design the infrastructure such that congestion is minimized, or in other words, the traffic volume should be controlled to be distributed best possible within the road network. Studies have shown [4, 90] that traffic in high congested networks produces approximately 50% more carbon-dioxide and other exhaust gases per vehicle and kilometer than in non-congested networks. Additionally, high congested roads cause air pollution by emitting carbon monoxide, volatile organic compounds, hydrocarbons, nitrogen oxides, particulate matter and other pollutants [27]. As high congestions mainly occur in metropolitan areas this has a huge negative effect on the traffic users and residents, and it has been shown that the morbidity rate for respiratory and cardiovascular diseases, cancer and adverse pregnancy is increased in such areas [59]. Additionally, highly congested roads also mean longer travel times and higher stress levels of commuters and other traffic users.

In conclusion, it is in society's best interest to minimize congestion, especially in metropolitan areas, in order to decrease greenhouse gas emissions and increase health and welfare.

Understanding traffic. In order to achieve these goals science has to take the first step and develop concepts and ideas on how to improve the infrastructure and reduce traffic congestions. But this is a challenging task as traffic dynamics are hard to predict and changes in the network can have surprising unintentional effects. For example the congestion in a network can increase when a new road is added or a bottle-neck road is enlarged even though the demand and total traffic volume has not changed. This famous effect is known as *Braess Paradox* [8, 9, 74] and was reported for example in Stuttgart in 1969 [52]. The reverse effect, i.e., a closure of a road that improves the traffic situation, was observed in New York (1990) [57] and Seoul (2005) [97]. Hence, it is important to understand the complicated interplay of traffic users and to predict it as best as possible before applying costly and sometimes irreversible changes to the infrastructure.

This is exactly the part where mathematics and this thesis come into play. In order to understand the traffic dynamics mathematicians need to formulate a strong traffic model, in which, on the one hand, it is possible to provide provable statements and predictions and which, on the other hand, represents real-world traffic in the best possible way.

Of course, different aspects of traffic have been considered from the mathematical perspective for a long time and there are several different approaches for modeling traffic [22, 23, 66, 69, 100]. But most of them are either too simplified, leading to large discrepancy to real-world traffic, or they are too complicated to prove any results for larger networks. Only in recent years there was a significant scientific improvement by combining the main concepts of previous static models with a continuous time approach [54].

In almost every case the traffic is modeled as a network flow on a graph and the traffic assignment problem is to find a traffic flow, for which every part that corresponds to a traffic user is traveling on a reasonable path from its origin to its destination. In simpler models time is not taken into account but instead it is assumed that the flow represents the average traffic on a constant demand. On the other side of the spectrum, we have models that aim to simulate the dynamics on each road segment in great detail including acceleration, deceleration, reaction times and distances between vehicles. In this thesis we consider a model that sits in the middle. It has a continuous time component in order to represent the dynamic flow evolution, as some of the most interesting phenomena can only be observed in over time models. But at the same time we stay on a macroscopic level and assume a straightforward queuing model on each link, which does not take detailed vehicle dynamics into account.

The overall goal of this thesis is to provide a complete overview of recent results on dynamic equilibria in this model and to extend the flow dynamics by features, such as spillback, kinematic waves or time-dependent capacities, to be as realistic as possible without losing the fundamental structure and results of dynamic equilibria.

In order to precisely formulate this traffic model the combination of multiple different mathematical disciplines is needed, each covering a different aspect. First of all, road networks are best represented by directed graphs, which are part of discrete mathematics. Next, game theory is important as we need to study the complex interplay of the single road users, which can all be seen as a huge number of players striving for the shortest travel time on the same network. Furthermore, the traffic simulation and congestion calculation form a dynamic system with continuous time, which requires several aspects of functional analysis and measure theory. Finally, if we want to predict the traffic behavior of larger instances, we need to implement an algorithm. It turns out that this can be done with mixed integer programming, a powerful tool within the area of combinatorial optimization.

1.1 Contribution and Overview

In this thesis we consider dynamic equilibria in a flow over time model with deterministic queuing, which were introduced by Koch and Skutella in 2009 [54]. This model, in its base version, lacks essential aspects of traffic. For example, each arc can store an unbounded amount of flow volume and, more importantly, only a single commodity with one source and one sink is considered. In other words, only scenarios where all agents share the same origin and the same destination can be modeled. As this is highly unrealistic for a real-world scenario, the goal of this thesis is to improve the base model by adding features, like spatial queues with kinematic waves that can cause spillback, or by allowing the network properties to change over time. We intensively study a multi-commodity version of this model and show the existence and some structural results of dynamic equilibria. Additionally, we consider a different equilibrium notation, where every agent adaptively chooses a fastest route to the destination based on the information that can be observed at the current time. Overall, the main contribution of this thesis is to greatly reduce the gap between mathematical Nash flows over time and complex large-scale simulation tools, such as MATSim [46], which only determine user equilibria heuristically without provable quality but which are used to predict traffic in real-world scenarios due to their large range of features.

Overview. First, we will present the state of the art and **related work** on this topic in the mathematics, computer science and traffic science communities in the next section.

Right after this, we are going to present all fundamental mathematical definitions, theorems and further **preliminaries** that are needed throughout this thesis in Chapter 2. This includes graph and game theory, an introduction into the Lebesgue measure and a brief overview of variational inequalities. We also consider basic structures of classical static flows as well as several optimization problems of flows over time.

The first chapter on Nash flows over time, Chapter 3, is dedicated to the **base model** as it was introduced by Koch and Skutella. Note that we changed some of the notation in order to make it consistent with the feature extensions in later chapters. In addition to the central results, i.e., that Nash flows over time can be constructed via a sequence of thin flows with resetting, we also provide a short overview of recent results about the long-term behavior and the price of anarchy in this model.

In Chapter 4 we consider flows over time in **time-dependent networks**. In other words, the capacities and the transit times (here given indirectly by speed limits) of the road network can change during the evolution of the traffic flow. If these changes are known information in advance and the traffic users, represented by flow particles, can anticipate them, then it is possible to construct a dynamic equilibrium. The key step is to show how to incorporate the time-dependent capacities and speed limits into the thin flow definition. This way we can prove that all main results of the base model also hold for this extension and that Nash flows over time can, again, be constructed by a sequence of these extended thin flows.

The biggest drawback of the base model is that we can only consider a single-commodity, i.e., only networks with a single source and a single sink, which corresponds to a dynamic traffic assignment problem, where each road user starts from the same origin and everyone has the same destination. To attack this flaw, we consider **multi-commodity flows over time** in Chapter 5, where each commodity has its own origin-destination pair. Even though we lose a lot of structure and some of the main results of the base model in this setting, we show that there still exist dynamic equilibria and that they can still be characterized by some extended thin flow formulation. The key idea is to take all flow from the past and the future into consideration at once and to incorporate the flow of other commodities, called *foreign flow*, into the thin flow definition. It is then possible to prove that multi-commodity Nash flows over time correspond one-to-one to multi-commodity thin flows. Finally, the existence of these thin flows can be shown by a reformulation to an infinite-dimensional variational inequality and the existence theorem of Brézis. As an additional result we show that all properties of the base model translate to the multi-commodity case, as long as all commodities either share the same origin but have different destinations or have potentially different origins but share the same destination.

Chapter 6 is dedicated to the most significant changes of the base model, namely **spillback** and **kinematic waves**. In order to represent highly congested road networks it is important to support these two crucial features. Spillback can be modeled by restricting the total amount of flow on an arc by some storage capacity. Whenever an arc is *full* the inflow rate cannot exceed the outflow rate any longer. In the words of traffic: if a road is full no new vehicle can enter the street before another vehicle has left. If more flow aims to use a full arc, it has to queue up on a previous arc. This is exactly the definition of spillback. In order to obtain a kinematic wave model, we introduce a backwards moving flow over time on each arc representing the gaps between vehicles. Whenever flow leaves an arc, it takes some time until these free spaces reach the tail, but only then, new flow is allowed to enter if this arc was full. Additionally to the introduction of these new features to the flow over time model, we show that essentially all results of the base model, especially the existence of dynamic equilibria, transfer to this extended model. The key idea thereby is to extend the thin flow definition by a so-called *spillback factor* for each node, which can be used to reduce the effective outflow capacity of all incoming arcs, whenever there is an outgoing arc that is fully congested. Unfortunately, most of the proof techniques used for the base model fail for these extensions. Hence, even though most of the results are basically the same as for the base model, the proofs are much more involved in this chapter.

Note that all these different extensions to the base model can also be combined in a straightforward manner. But in order to keep things as simple as possible, we focus on only one extension to the base model in each chapter.

We continue with a different equilibrium concept in Chapter 7. Here, we consider the same multi-commodity flow over time model as given in Chapter 5 but now the particles, representing the road users, are not allowed to anticipate the further evolution of the flow, but instead, they consider only the current state of the network in order to decide on a route to their destination. In these **instantaneous dynamic equilibria**, or **IDE flows** for short, each particle can reevaluate its route choice at every node and adapt its path if necessary. The advantages and disadvantages of this concept compared to Nash flows over time are discussed in-depth and even though these flows over time have completely different structures, we can still use the general thin flow framework to compute them. In fact, the existence of multi-commodity IDE flows can be shown with techniques very similar to the ones used for proving the existence of single-commodity Nash flows over time.

Finally, in Chapter 8 we give a brief summary of the results and discuss **further research** as well as several open problems related to Nash flows over time.

How to read this thesis. The goal of this thesis is that everyone with a basic understanding of mathematical notation should in principle be able to comprehend all results and proofs presented here. Furthermore, all lemmas and theorems that are either a central result on their own or that are necessary later on, are proven rigorously, with the exception of a few theorems in the preliminaries,

for which the proofs are beyond the scope of this thesis. In these cases we refer to an article or text book, where the respective proof can be found.
Each of the Chapters 4 to 6, which focus on a special extension aspect, are completely independent of each other and only refer to the base model. In other words, it is possible to read only Chapter 3 and then one of the Chapters 4 to 6. To understand Chapter 7 about IDE flows it is helpful to read the introduction of the multi-commodity flow over time model in Section 5.1.1.
Note that the fundamental philosophy of this thesis is to keep everything as simple and comprehensible as possible, without losing any mathematical precision or correctness. Surely, the proofs will also include some unpleasant technicalities and calculations that are unavoidable. But we aim to always convey the intuitive idea behind all concepts and proofs. In order to keep the reading flow as fluid as possible we move some of the especially technical and strenuous proofs to the appendix of the corresponding chapter and replace them by intuitive presentations of the key ideas.

1.2 Related work

Dynamic traffic assignment. Network loading models in time-dependent networks have been studied intensively within the traffic science community and the several different approaches can be classified into three categories depending on their level of detail. On the **macroscopic** level, traffic is modeled by a flow representing a collection of vehicles. Some of the pioneer work in this regard is due to Vickrey with his single link-load model [96] and to Mechant and Nemhauser with an exit-function-based flow model [67]. Recently, more advanced approaches like the Colombo phase transition model [6, 16] or the macroscopic node model [31, 92] have become popular. In contrast to this, **microscopic** traffic flow models consider each vehicle individually and track not only the position, speed and acceleration of each car, but they also simulate maneuvers, like lane changes and overtaking. Some of them even consider different driver behavior, such as gap-acceptance, reaction times and more. For further details on this kind of models we refer to the comprehensive surveys of Algers et al. [1] and Olstam and Tapani [70]. Finally, **mesoscopic** models are in between these two. Some aspects, such as the traffic dynamics, are considered at a low level of detail (macroscopic) whereas other aspects, such as the agent behavior and the route choice for each traffic user, are considered individually (microscopic). This gives a trade-off between accuracy and computational complexity. Examples of such models are DynaSMART [47] and the multi-agent transport simulation (MATSim) [46]. For a more detailed overview of different dynamic traffic assignment models we refer to the book of Ran and Boyce [73] and the survey article of Wang et al. [98].

Flows over time. Classical network flows, or **static flows** as we call them in this thesis, have been studied from a mathematical perspective since the middle of the last century. Ford and Fulkerson did a lot of pioneer work on these structures and they also were the first to introduce **dynamic flows**, also called **flows over time**, back in 1958 [32, 33]. In a flow over time, every flow particle travels over time through a network, and therefore, it is an excellent basis for a traffic model. Considering a network with a single source and a single sink as well as a capacity and a transit time for each arc, Ford and Fulkerson showed how to efficiently construct a **maximum flow over time** for some given time horizon. Hereby, a maximum flow over time is a flow over time that sends as much flow volume as possible from the source to the sink within the time horizon. Their algorithm is based on a static min-cost flow computation in the given network, where arc transit times are interpreted as costs. The resulting static flow corresponds to the flow rates of a maximum flow over time that needs to be sent into the network and along the paths as long as possible.

Closely related to the maximum flow over time problem is the **quickest flow problem**. Here, a specific amount of flow volume is given and the task is to send all of it as quickly as possible to the sink. This can be solved efficiently by using Ford's and Fulkerson's algorithm in combination with a binary search framework [29], which can be improved to a strongly-polynomial running-time by using parametric search [11].

Surprisingly, it is possible, at least for the single-source single-sink setting, to compute a flow over time that is maximal for all time horizons simultaneously, and that is, therefore, also a quickest flow for all given flow volumes at once. Such special flows are called **earliest arrival flows** and their existence was already shown by Gale in 1959 [35]. Minieka showed in 1973 that they can be computed by using the successive shortest path algorithm, where it is also allowed to send negative flow backwards in time in the opposite direction of an arc [68]. Unfortunately, this might take an exponential number of iterations in general, which was shown by Zadeh [101]. Just as for all other flow over time concepts, earliest arrival flows were first considered in a discrete time model and, only in 1990, Philpott [72] showed their existence in the continuous time model. Eight years later Fleischer and Tardos [29] also extended the earliest arrival algorithm to the continuous time setting. When considering networks with several sinks, it is possible that there does not exist an earliest arrival flow anymore [30]. In fact, it is NP-hard to decide whether such a flow exists or not in these networks [84]. A formal definition of all of these flow over time problems is given in Section 2.7 on page 24.

It turns out that the **transshipment over time problem** of balancing given supplies and demands on different nodes of a network by sending flow within a time horizon is considerably more difficult to solve than the corresponding problem for static flows. Only in 2000, Hoppe and Tardos [45] presented an efficient algorithm for solving the transshipment over time problem; see also Hoppe's PhD thesis [44]. However, their algorithm uses parametric submodular function minimization, which is theoretically efficient but leads to unrealistic running times for reasonably large networks in practice. Only quite recently, Schlöter and Skutella [83] presented an improvement of this result; see also Schlöter's PhD thesis [82].

As dynamic flow models were often considered in a discrete time model, a classical approach to solve flow over time problems is to reduce them to a static flow problem by considering the **time-expanded network**. The original network is for this purpose copied for every time step between zero and the time horizon, and for each original arc the tail-node in each copy is connected to the head-node in the respective copy corresponding to a later point in time depending on the transit time. This way, an optimal static flow in the expanded network corresponds to an optimal flow over time in the original network. However, the size of the expanded network is exponential in the input size under the natural assumption that the time horizon is encoded in binary. This construction was already introduced by Ford and Fulkerson [32, 33] and it is used, for example, to prove that the **minimum cost flow over time problem**, which is NP-hard in general [51], can be solved in pseudo-polynomial time. Moreover, clever usage of the concept of time-expanded networks was utilized by Fleischer and Skutella to show the existence of fully polynomial-time approximation schemes for several NP-hard flow over time problems [28]. Another clear evidence that flow over time problems are more complex than static flow problems is the fact that the computation of multi-commodity flows over time is NP-hard [38]. For a recent introductory survey into the whole field of flows over time we refer to the publication of Skutella [89]. Furthermore, the survey of Köhler, Möhring and Skutella gives an overview of flow over time models in the context of traffic networks [56].

Nash flows over time. All flow over time problems discussed so far are based on the assumption that the total flow is controlled by a central authority, which decides on the route and departure time of each single particle. In real-world traffic situations, however, each traffic user acts independently and selfishly, and therefore, we have a lack of coordination. To capture this behavior, we assume that each flow particle is an individual agent that wants to reach its destination as early as possible, and hence, we consider flows over time from a game-theoretical perspective. In this thesis we study dynamic equilibria, which are states where no particle can reach the destination earlier by unilaterally changing its route. Hereby, the arc dynamics are described by the **deterministic queuing model**, which was first mentioned by Vickrey in 1969 [96] and studied by Hendrickson and Kocur in 1981 [41]. It is also at the core of large-scale agent-based traffic simulations, such as MATSim developed by Horni, Nagel, Axhausen [46] and others. Here, the travel time of a flow particle entering an arc consists of a constant transit time and a waiting time due to a queue that builds up in front of a bottle-neck whenever the flow rate exceeds the arc's capacity. The queues, and therefore the arc dynamics, thereby follow the first-in-first-out (FIFO) principle. For a detailed definition we refer to Section 3.2.

In 2009, Koch and Skutella characterized the structure of dynamic equilibria in a single-source single-sink network from a strictly mathematical point of view [54]; see also Koch's PhD thesis [53]. As the most essential structural insight, they prove that these equilibrium flows, called **Nash flows over time**, consist of a number of phases, in which all flow entering the network chooses the same routes from the source to the sink. Each phase is, thereby, characterized by the strategy of the particles in form of static flows featuring specific properties, which they call **thin flows with resetting**. Based on this key observation, Cominetti, Correa and Larré showed existence and uniqueness of these thin flows with resetting [18], and thus proved the existence of Nash flows over time. They extended this existence result in 2015 to networks with general inflow rate functions and also to a multi-commodity setting [17], which we will discuss in Chapter 5. Moreover, Macko, Larson and Steskal showed the existence of the Braess Paradox in this model [65], and Cominetti, Correa and Olver examined the long-term behavior of queues and were able to bound their lengths whenever the network capacity is sufficiently large [19]. Nash flows over time in the deterministic queuing model are the central mathematical object considered in this thesis. More details on most of these results and a formal introduction to the model can be found in Chapter 3.

Price of anarchy. The price of anarchy is a concept in game theory to measure the total loss in a game due to the selfish behavior of the players. It is defined as the ratio of the total cost of the worst Nash equilibrium and the total cost in the optimal (cooperative) scenario. This concept was first considered by Dubey and Jonathan [25] and transferred to static routing games by Roughgarden and Tardos [77, 75]; see also the publication of Koutsoupias and Papadimitriou [58]. One possibility to apply this concept to the flow over time model is to measure the increase of the arrival time of a particle in a Nash flow over time compared to the arrival time of the same particle in an earliest arrival flow. Bhaskar, Fleischer and Anshelevich showed that this ratio can be bounded by $\frac{e}{e-1}$ under some very specific conditions on the network [5]. Only very recently, Correa, Cristi and Oosterwijk reduced these preconditions significantly [20]. But the conjecture, that the price of anarchy is $\frac{e}{e-1}$ in general, remains open. More on this topic can be found in Section 2.3 on page 14 for static flows and in Section 3.6.3 on page 41 for flows over time in the base model.

Wardrop equilibria. User equilibria have also been studied for a long time in static networks and they are referred to as **Wardrop equilibria** [99]. As they lack the time component, they can be thought of as a steady state after an initial evolution when considering a constant incoming traffic rate [61]. Here, the network is considered to have monotonically increasing cost functions that map the load of each arc, i.e., the amount of flow using this arc, to the traversing time every traffic user experiences by using this arc. Then, the Wardrop equilibrium simply is a static flow where each arc that is used by a positive amount of flow lies on a path from the origin to the destination with minimal cost (a formal definition can be found in Section 2.3 on page 14). These flows always exist [2] and are unique whenever the cost functions are strictly increasing [91]. Under some mild conditions on the cost functions they can be computed efficiently via convex optimization [2, 23]. Just recently, Klimm and Warode showed that it is even possible to compute all Wardrop equilibria for a fixed network with piece-wise linear cost functions for all possible demands [50]. Roughgarden studied the price of anarchy in this model and showed that it only depends on the class of cost functions and not on the network topology [76]. Together with Tardos he showed that the price of anarchy is $4/3$ for linear cost functions [77]. For more details on this topic we refer to the survey of Correa and Stier-Moses [21].

Packet routing. In all the approaches presented above we model traffic by a flow over time or a static flow that can be split in arbitrarily small particles, which makes sense when considering traffic from a microscopic point of view. But there is another class of mathematical routing models that considers each vehicle separately. This perspective became popular since Leighton, Maggs and Rao [62] studied optimization problems in such **packet routing models**. Combining these with game theory leads to so-called **competitive packet routing models**, or **routing games with atomic players**, where discrete packets (corresponding to the vehicles or players) want to move over time as fast as possible through the network to their destinations. Whenever the number of packets using an arc exceeds the arc's capacity, some of the players have to wait, building up congestion. These models are widely studied and there are different aspects that can be altered. For example, Hoefer et al. and Kulkarni et al. study versions with continuous time [43, 60], whereas Harks et al. consider discrete time steps [40]. A model where each player controls multiple packets at the same time was recently introduced by Peis et al. [71] and a competitive packet routing model, which is very similar to a discrete version of Nash flows over time, is studied by Scarsini et al. [81].

2 Preliminaries

In this chapter we give a formal introduction of all mathematical objects and concepts required throughout this thesis. We start with the basics for graphs, networks and static flows in Sections 2.1 and 2.2 and continue with the fundamental definitions for game theory and a brief introduction into Wardrop equilibria and the price of anarchy in Section 2.3. Integrable functions are essential for flows over time and they are introduced together with the basics of differentiation and measure theory in Section 2.4. Advanced results of finite and infinite dimensional variational inequalities and the fixed point theorem of Kakutani are presented in Section 2.5. They are required for proving the existence of thin flows as well as the existence of Nash flows over time in a multi-commodity setting. Section 2.6 is dedicated to a brief introduction into linear and integer programming, which we need to characterize the long-term behavior of Nash flows over time and to compute thin flows. Finally, we define flows over time in Section 2.7 and show some basic results about optimal flows over time in order to set dynamic equilibria into context. At the very end of this chapter in Section 2.8 we give an intuition of the notation we use throughout this thesis.

2.1 Graphs and Networks

Directed graphs. A **directed graph**, here also just called **graph**, $G = (V, E)$ is given by a finite **node set** V and an **arc set** $E \subseteq V \times V \setminus \{ (v, v) \mid v \in V \}$ as it is depicted in Figure 2.1. Note that by this definition, we do not allow graphs to have any parallel arcs or loops.
Each **arc** $e = (u, v) \in E$ is imagined as a directed arrow pointing from u to v and, for the sake of better readability, it is denoted by uv. We call u the **tail** and v the **head** of arc e and in rare occasions we write $\mathrm{tail}(e)$ for u and $\mathrm{head}(e)$ for v. The sets of all **incoming** and **outgoing** arcs of a node v are denoted by

$$\delta_v^- := \{ e \in E \mid \mathrm{head}(e) = v \} \qquad \text{and} \qquad \delta_v^+ := \{ e \in E \mid \mathrm{tail}(e) = v \}.$$

For a subset of nodes $W \subseteq V$ we write

$$\delta_W^- := \{ e = uv \in E \mid u \notin W, v \in W \} \qquad \text{and} \qquad \delta_W^+ := \{ e = uv \in E \mid u \in W, v \notin W \}.$$

The **in-degree** and **out-degree** are given by $|\delta_v^-|$ and $|\delta_v^+|$ and the (total) **degree** by $|\delta_v^-| + |\delta_v^+|$.

Paths and Cycles. Given a directed graph $G = (V, E)$ a finite sequence of arcs $(e_1, e_2, \dots, e_m)$ is called a **path** of **length** m if $\mathrm{head}(e_i) = \mathrm{tail}(e_{i+1})$ for $i = 1, 2, \dots, m-1$. Paths can also be denoted by the visited nodes, i.e., a sequence of nodes $P = (v_0, v_1, \dots, v_m)$ is a path of length m if $v_{i-1}v_i \in E$ for all $i = 1, 2, \dots, m$. Note that the empty sequence is a path of length 0. We say a node v is **visited** by a path P if it occurs in the node representation of the path, in this case we also write $v \in P$. Similarly, we say P contains an arc e if e occurs in the arc representation of P and we write $e \in P$.
A **cycle** of length m is a path $C = (e_1, e_2, \dots, e_m)$ with $\mathrm{head}(e_m) = \mathrm{tail}(e_1)$, or equivalently, a path $C = (v_0, v_1, \dots, v_m)$ with $v_m = v_0$. A graph is called **acyclic** if it does not contain any cycles. Note that the empty sequence is explicitly not a cycle.

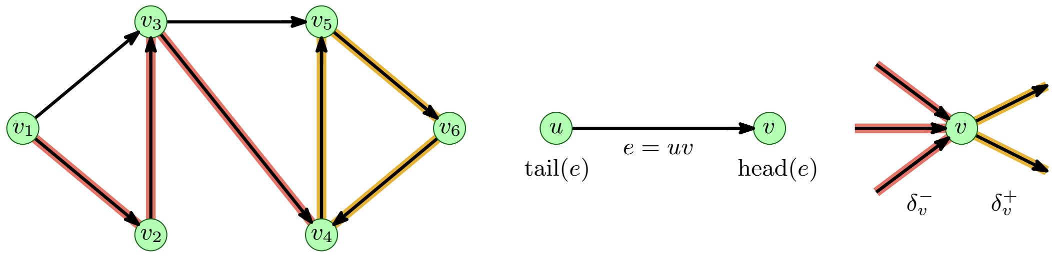


Figure 2.1: *On the left:* a directed graph with six nodes and eight arcs. The node sequence (v_1, v_2, v_3, v_4) forms a simple path and (v_4, v_5, v_6, v_4) is a simple cycle. The graph is weakly connected but not strongly connected as, for example, v_1 is not reachable by v_2. *In the middle:* an arc e with tail u and head v. *On the right:* A node v with incoming arcs δ_v^- and outgoing arcs δ_v^+.

We say a node u can **reach** a node v (or v **is reachable** by u) in G if there exists a u-v-path, i.e., a path starting at u and ending at v. Furthermore, a graph G is (weakly) connected if for every pair of nodes $u, v \in V$ either v is reachable by u or u is reachable by v. If we have a graph G such that for every pair of nodes $u, v \in V$ it holds that u can reach v and v can reach u, we say that G is strongly connected.

We call a path $(v_0, v_1, \dots, v_m)$ **simple** if none of the nodes are visited twice, in other words, if $|\{ v_0, v_1, \dots, v_m \}| = m + 1$. A cycle is called **simple** if no node is visited twice, except for the start and end node, which of course need to be the same.

Networks. As **networks** we usually denote a directed graph together with some distinct nodes, parameters on the arcs, and additional objects defining the setting. Note that the definition of a network depends on the context. For flows over time a network often consists of a directed graph G together with arc capacities $\nu_e > 0$ and transit times $\tau_e \geq 0$ for all $e \in E$ as well as two distinct nodes $s, t \in V$ and a network inflow rate $r > 0$. Hence, at least for the base model the network is defined as the tuple $(G, (\nu_e)_{e \in E}, (\tau_e)_{e \in E}, s, t, r)$. Later on some of these constants may get replaced by functions (see time-dependent networks in Chapter 4) or expanded by additional parameters. For the kinematic wave model, for example, we also have an inflow capacity, a storage capacity and a gap transit time for each arc. For every model we clearly state the properties of the network at the beginning of the corresponding chapter.

2.2 Static Flows

We will call the classical network flows *static flows* in this thesis in order to dissociate them from *dynamic flows* (here called *flows over time*). Given a network consisting of a directed graph G with source s and sink t, as well as a capacity $\nu_e > 0$ on each arc, we call an vector $x = (x_e)_{e \in E}$ of **arc flow values** $x_e \geq 0$ a **static flow** if it satisfies **flow conservation** at each node $v \in V \setminus \{ s, t \}$:

$$\sum_{e \in \delta_v^-} x_e = \sum_{e \in \delta_v^+} x_e.$$

The **flow value** $|x|$ is hereby defined as the net flow leaving the source (or equivalently entering the sink) and we require it to be non-negative:

$$|x| := \sum_{e \in \delta_s^+} x_e - \sum_{e \in \delta_s^-} x_e = \sum_{e \in \delta_t^-} x_e - \sum_{e \in \delta_t^+} x_e \geq 0.$$

Furthermore, we call a static flow **feasible** if the flow on each arc does not exceed the capacity, i.e., if $x_e \leq \nu_e$ for all $e \in E$.

Path-based static flows. Instead of specifying a static flow by the flow value on each arc, it is also possible to describe it path-based. For this let $\mathcal{P}$ be the set of all simple s-t-paths and $\mathcal{C}$ the set of all simple cycles in the graph G. Note that both of these sets are finite since E is finite. We call a vector $(x_P)_{P \in \mathcal{P} \cup \mathcal{C}}$ with $x_P \geq 0$ also **static flow** with **flow value** $\sum_{P \in \mathcal{P}} x_P$.
To convert a path-based static flow into an arc-based static flow we can simply determine the arc loads in the following way:

$$x_e := \sum_{P \ni e} x_P.$$

Here the term "$P \ni e$" denotes all simple paths and simple cycles that contain arc e.

Static flow decomposition. The inverse direction follows by the famous flow decomposition theorem:

Theorem 2.1.
Given an arc-based static flow $x = (x_e)_{e \in E}$ it is possible to specify a path-based static flow $(y_P)_{P \in \mathcal{P} \cup \mathcal{C}}$ with

$$x_e = \sum_{P \ni e} y_P.$$

Proof. The key idea is to start with the static flow $x^1 := x$ and to consider one path/cycle $P \in \mathcal{P} \cup \mathcal{C}$ after the other. In each iteration we set y_P to the minimal flow value of all arcs in P, i.e., $y_P := \min_{e \in P} x_e^1$. Afterwards, we reduce the flow of all $e \in P$ by this value, while keeping the flow of all other arcs as they were, i.e., $x_e^2 := x_e^1 - y_P$ for $e \in P$ and $x_e^2 := x_e^1$ for $e \notin P$.
We show that we end up with the empty static flow x^k, meaning $x_e^k = 0$ for all $e \in E$. Suppose there is an $e_1 = v_0 v_1 \in E$ with $x_{e_1}^k > 0$. In the case of $v_1 \neq t$ we find an arc $e_2 = v_1 v_2 \in \delta_{v_1}^+$ with $x_{e_2}^k > 0$ due to flow conservation. We can continue this argumentation until we have a node v_m with $v_m = v_j$ for some $j < m$ or we have $v_m = t$. In the first case we found a simple cycle $C = (v_j, v_{j+1}, \ldots, v_m)$ with positive flow on each arc. This is a contradiction, as we would have reduced this flow when considering C. In the second case, we can use the flow conservation to build a path backwards starting at v_0, where we will end up at the source s (or, again, have a simple cycle). Hence, if we end up at s we found a simple s-t-path P with positive flow on each arc, which is again a contradiction as we would have reduced this flow to 0 at some of these arcs when considering P. Since x^k is the empty static flow we have by construction that

$$0 = x_e^k = x_e - \sum_{P \ni e} y_P \qquad \text{for all } e \in E.$$

□

In the original statement of the flow decomposition theorem the number of paths P with $y_P > 0$ is bounded by $|E|$ and a path-based formulation can be computed efficiently by iteratively constructing paths for which the remaining flow value is strictly positive. Further details are omitted as these statements are not needed in the thesis.
Note, however, that in an acyclic graph it holds true that for every static flow x with flow value $|x|$ we have $x_e \leq |x|$ for all arcs e. This can easily be seen as we do not have any cycles in the path based formulation, and therefore,

$$x_e = \sum_{P \ni e} x_P \leq \sum_{P \in \mathcal{P}} x_P = |x|.$$

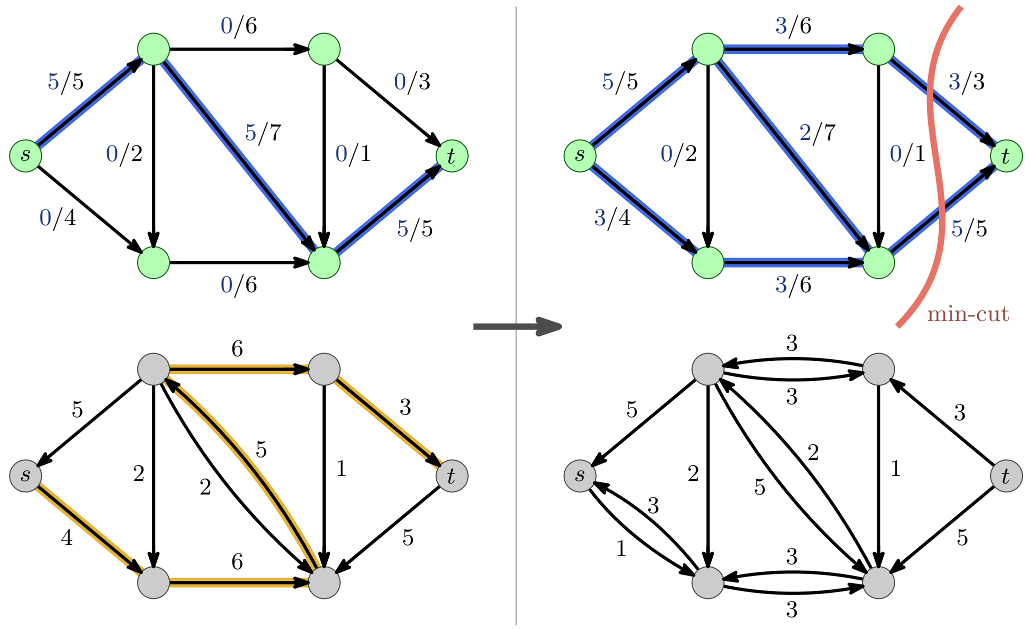


Figure 2.2: After sending 5 flow units along the first s-t-path (*top left*) the Ford and Fulkerson algorithm picks the next s-t-path in the residual network (*bottom left*). By sending 3 flow units along this augmenting path (*top right*) the algorithm stops since t is not reachable by s in the corresponding residual network (*bottom right*). The final static flow with flow value 8 is maximal as the min-cut also has a value of 8.

Cuts. For a given network $G = (V, E)$ with source s, sink t and arc-capacities ν_e we call the outgoing arcs $C = \delta_S^+$ of a set $S \subseteq V \setminus \{ t \}$ with $s \in S$ an s-t-**cut**. An s-t-cut always separates the source from the sink and every s-t-path has to use at least one arc of C. The **capacity** or **value** of an s-t-cut C is defined by $|C| \coloneqq \sum_{e \in C} \nu_e$.

Residual networks. Consider a feasible static s-t-flow $x = (x_e)_{e \in E}$ in a network with graph $G = (V, E)$, source s, sink t and capacities $(\nu_e)_{e \in E}$. Without loss of generality we assume that for every pair of nodes $u, v \in V$ only arc uv or arc vu is contained in E but not both. In a general graph this can be achieved by adding dummy nodes if necessary. The **residual network** for x is given by a directed graph $\bar{G} = (V, \bar{E})$ with

$$\bar{E} \coloneqq \{ e \mid e \in E \text{ with } x_e < \nu_e \} \cup \{ vu \mid uv \in E \text{ with } x_{uv} > 0 \}$$

and capacities $\bar{\nu}_e \coloneqq \nu_e - x_e$ and $\bar{\nu}_{vu} \coloneqq x_e$ for all $e = uv \in E$. Two examples of residual networks are displayed on the bottom left and bottom right of Figure 2.2. The arcs $e \in E \cap \bar{E}$ are called **forward arcs**, whereas the arcs e with $e \in \bar{E} \setminus E$ are called **backward arcs**. Finally, the s-t-paths in the residual network are called **augmenting paths**.

Optimal static flows. Given an s-t-network $G = (V, E)$ with capacities $(\nu_e)_{e \in E}$ such that t is reachable by s, the **static maximum flow problem** is to determine a feasible static s-t-flow of maximum value. Such a maximum static flow can, for example, be constructed by the **Ford and Fulkerson algorithm** by starting with the empty static flow and iteratively considering one augmenting path after the other in the residual network; see Figure 2.2. By augmenting the static flow by the minimal capacity along the considered path, meaning to increase the flow value on forward arcs and reducing it on backwards arcs, we can increase the flow value of the static flow. This procedure can be repeated until there is no augmenting path left in the residual network. By the well-known **max-flow**

min-cut theorem, that states that the value of a maximum static s-t-flow equals the capacity of a minimal s-t-cut, it can be shown that this is indeed a maximum static flow.
With a similar algorithm it is possible to determine a **static min-cost flow** in a network that, in addition, has a cost $c_e \geq 0$ on each arc. The task is to determine a feasible static flow that is not only a maximum flow but also has minimal cost, where the cost of the static flow x is given by $\sum_{e \in E} c_e \cdot x_e$. The **successive shortest path algorithm** works very similar to the algorithm of Ford and Fulkerson but instead of considering the augmenting paths in arbitrary order, it always augments a path of minimal cost, where the cost of each backwards arc vu is given by $-c_{uv}$.

Static transshipments. A generalization of a static s-t-flow is a static transshipment. For a transshipment network we consider a directed graph $G = (V, E)$ where every node v is equipped with a supply or demand of $b_v \in \mathbb{R}$. A positive value means that v has a supply and a negative value corresponds to a demand. We assume that the total supply equals the total demand, i.e., $\sum_{v \in V} b_v = 0$. We call an arc vector $(x_e)_{e \in E}$ with $x_e \geq 0$ a b-**transshipment** if

$$\sum_{e \in \delta_v^+} x_e - \sum_{e \in \delta_v^-} x_e = b_v \qquad \text{for all } v \in V.$$

These transshipments often come together with a cost $c_e \geq 0$ on each arc and a typical task is to find a transshipment of minimal cost $\sum_{e \in E} x_e \cdot c_e$.
Again, a static transshipment can be represented by a path-based formulation, where we consider a value for every simple path starting at some supply node and ending at a demand node together with all directed cycles.
It follows immediately by the path-based representation of the b-transshipment that in all acyclic graphs we have

$$x_e = \sum_{P \ni e} x_P \leq \sum_{P \in \mathcal{P}} x_P = \sum_{v \in V \,:\, b_v > 0} b_v. \tag{2.1}$$

This property of b-transshipment will be used in Chapter 6 for the kinematic wave model.

2.3 Game Theory

In this thesis we model the dynamic traffic assignment problem by a game where each traffic user is a player trying to minimize his or her arrival time at the destination. The main focus will be on Nash flows over time, which are the equilibria in this game. To provide a good intuition of this concept we will give a short introduction into games with a finite number of players, which will later be translated into dynamic games with a continuum of players.

Games. In mathematical game theory a **game** consists of a set of **players** N, a set of **strategies** X_i for each player $i \in N$ as well as a **payoff-function**

$$P \colon X \to \mathbb{R}^N, \qquad \text{where } X := \bigtimes_{i \in N} X_i.$$

For a given **strategy profile** $x = (x_i)_{i \in N} \in X$ we call $P_i(x)$ the **payoff of player** i. The goal of each player is to maximize his or her own payoff.
For a given strategy profile $x \in X$ and a player $j \in N$ we use the following unintuitive, but very useful, notation and write x_{-j} for the strategy vector $(x_i)_{i \in N \setminus \{\, j \,\}}$ and for a strategy $y \in X_j$ we write (x_{-j}, y) for the strategy profile that is obtained by replacing x_j by y in x.

Nash equilibria. The most important objects in game theory are equilibria, strategy profiles in a stable state, which means that no player can increase his or her payoff by changing his or her strategy. More precisely, we call a strategy profile x a **Nash equilibrium** if

$$P_i(x) \geq \max_{y \in X_i} P_i((x_{-i}, y)) \quad \text{for all } i \in N.$$

We denote the set of strategy profiles that forms a Nash equilibrium by NE.

Static network games. To give an example of a game with infinite many players we want to consider the **static traffic assignment problem**. The key idea is, that each road is modeled by an arc equipped with a cost function, which describes the delay for each traffic user on that road dependent on the amount of traffic using it. Note that this is a static game as we do not consider any time component. More formally, we consider a network consisting of a directed graph $G = (V, E)$ with a source s, a sink t as well as monotonically increasing cost functions $c_e := [0, \infty) \to [0, \infty)$. For some $a > 0$ let $N := [0, a]$ be a set of uncountably many players. The strategy space X is identical for each player and consists of all simple s-t-paths, i.e., $X := \mathcal{P}$. A strategy profile is given by a function $f\colon N \to \mathcal{P}$ mapping each player to its chosen route. For technical reasons we have to restrict the set of valid strategy profiles and require that this mapping is measurable (see Section 2.4.1). In other words, the sets $N_P := \{\, i \in N \mid f(i) = P \,\}$ have well-defined measures for all $P \in \mathcal{P}$. This way, each valid strategy profile corresponds to a static s-t-flow x of value a, which is obtained by setting x_P to the measure of N_P. The payoff in this game will be represented by a cost function c_i for each player i (corresponding to his or her travel time), which he or she tries to minimize. For a given static flow x the cost of a path P is given by $c_P(x) := \sum_{e \in P} c_e(x_e)$ and the cost of a player $i \in N$ equals the cost of the chosen path.

Wardrop equilibria. The corresponding static flows of Nash equilibria in this type of games are called **Wardrop equilibria**, as they were first considered by Wardrop in 1952 [99]. The Nash condition translates to the following defining condition of Wardrop equilibria:

$$c_P(x) \leq \min_{Q \in \mathcal{P}} c_Q(x) \quad \text{for all } P \in \mathcal{P} \text{ with } x_P > 0.$$

Clearly, this condition guarantees that every path that is used by a positive measure of players is a fastest s-t-path when interpreting the costs as delay. In other words, no player has the incentive to choose a different route. An example of a Wardrop equilibrium is depicted in Figure 2.3. As a central result, it has been shown that these Wardrop equilibria always exist and, in the case that all cost functions are strictly increasing, that they are unique [2].

Price of anarchy and price of stability. The **price of anarchy** is a concept to measure the loss of total payoff when every player acts selfishly compared to the total payoff if all players cooperate. More precisely, it is the quotient of the sum of all payoffs in an optimal (cooperative) state and the sum of all payoffs in the worst Nash equilibrium

$$\text{PoA} := \frac{\max_{x \in X} \sum_{i \in N} P_i(x)}{\min_{x \in \text{NE}} \sum_{i \in N} P_i(x)} \geq 1.$$

If, instead of payoffs, we consider costs that the players try to minimize, the price of anarchy is instead defined by

$$\text{PoA} := \frac{\max_{x \in \text{NE}} \sum_{i \in N} P_i(x)}{\min_{x \in X} \sum_{i \in N} P_i(x)} \geq 1.$$

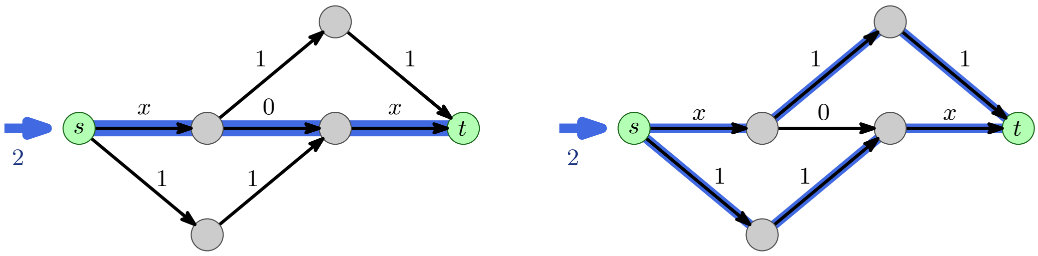


Figure 2.3: Static traffic assignment in a network, where the first and last arc on the middle path have a cost function of $c_e(x) = x$ and all other costs are constant 1 or 0. *On the left:* The Wardrop equilibrium, where both flow units take the middle path with cost of 4 giving a total cost of 8. As all other paths also have a total cost of 4 there is no incentive for flow particles to deviate. *On the right:* The optimal static flow. Here one flow unit takes the top and one unit the bottom path. The total cost is therefore 6. Hence, since the Wardrop equilibrium is unique, the price of anarchy and stability in this network is $8/6$. Note that removing the middle arc with cost 0 would force the Wardrop equilibrium to be optimal. This Braess Paradox is the reason why this specific network is called **Braess network**.

If we consider a class of games, for example, all static traffic assignment problems with affine linear cost functions, the price of anarchy of the class is defined as the supremum of the prices of anarchy of all instances.

A related concept is the **price of stability**. Here, we compare the optimal value to the best Nash equilibrium instead of the worst. In other words, we suppose that an authority can force every player to choose a specific strategy but the outcome will only persist if no player has the incentive to change his or her strategy. Then the price of stability is a measure of efficiency for these stable states compared to the optimum. Hence, it is defined as

$$\mathrm{PoS} := \frac{\max_{x\in X} \sum_{i\in N} P_i(x)}{\max_{x\in \mathrm{NE}} \sum_{i\in N} P_i(x)} \geq 1$$

or, if we have a minimizing game, as

$$\mathrm{PoS} := \frac{\min_{x\in \mathrm{NE}} \sum_{i\in N} P_i(x)}{\min_{x\in X} \sum_{i\in N} P_i(x)} \geq 1.$$

The price of anarchy for static traffic assignment games has been extensively studied by Roughgarden and others [77, 76, 75] for different classes of cost functions. For affine linear functions and arbitrary networks the price of anarchy is, for example, $4/3$ [77]. Hence, the example given in Figure 2.3 is a worst-case instance.

2.4 Functions

As we consider dynamic traffic assignment problems the traffic flow changes over time, which is modeled by functions that specify the flow rates at any given point in time. In this section we want to briefly recall the most important properties of functions on the reals.

Basic properties. Consider a function $f\colon D \to \mathbb{R}$ with a domain $D \subseteq \mathbb{R}$. We call f **bounded** if there exists a bound $B \in \mathbb{R}$ with $|f(\theta)| \leq B$ for all $\theta \in D$ and f is called **right-constant** if for every $\theta_0 \in D$ there exists an $\varepsilon > 0$ with $f(\theta_0) = f(\theta)$ for all $\theta \in [\theta_0, \theta_0 + \varepsilon] \cap D$. Furthermore, we call f **(monotonically) increasing** if $f(\theta_1) \leq f(\theta_2)$ for all $\theta_1 < \theta_2$ and **(monotonically) decreasing** if $f(\theta_1) \geq f(\theta_2)$ for all $\theta_1 < \theta_2$ with $\theta_1, \theta_2 \in D$. It is strictly increasing (or strictly decreasing) if the corresponding inequality is **strict**. The function f is **continuous at a point** $\theta_0 \in D$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ with

$$|f_e(\theta_0) - f_e(\theta)| < \varepsilon \qquad \text{for all } \theta \in [\theta_0 - \delta, \theta_0 + \delta] \cap D.$$

A function f is called **continuous** if it is continuous at all points in D and it is called **L-Lipschitz continuous** if there exists a constant $L \geq 0$ such that

$$|f_e(\theta_1) - f_e(\theta_2)| \leq L \cdot |\theta_1 - \theta_2| \qquad \text{for all } \theta_1, \theta_2 \in D.$$

One of the most important properties of continuous functions is described by the following well-known theorem:

Theorem 2.2 (Intermediate value theorem).
Considering a continuous function $f\colon I \to \mathbb{R}$ defined on a closed interval $I := [a, b] \subseteq \mathbb{R}$, for every value $\lambda \in [f(a), f(b)]$ there exists a $\xi \in [a, b]$ with $f(\xi) = \lambda$.

A proof for this theorem can be found in most text books about basic calculus, for example in the book by Royden and Fitzpatrick [78].

Differentiation. Suppose that the domain D is an open set. Then f is called **right-differentiable** at $\theta_0 \in D$ if the following limit, called **right-derivative**, exists:

$$\lim_{\theta \searrow \theta_0} \frac{f(\theta_0) - f(\theta)}{\theta_0 - \theta}.$$

Here, $\theta \searrow \theta_0$ means that θ goes to θ_0 while always staying strictly larger, i.e., $\theta \to \theta_0$ with $\theta > \theta_0$. Analogously, we call f **left-differentiable** at $\theta_0 \in D$ if the following **left-derivative** exists:

$$\lim_{\theta \nearrow \theta_0} \frac{f(\theta_0) - f(\theta)}{\theta_0 - \theta}.$$

If f is both right- and left-differentiable at $\theta_0 \in D$ and the left and right derivative are equal, we call f **differentiable at** θ_0. In this case the derivative value is denoted by $\frac{\mathrm{d}f}{\mathrm{d}\theta}(\theta_0)$ or by $f'(\theta_0)$. Finally, f is called **differentiable** if it is differentiable at all $\theta_0 \in D$. All derivative values together form the **derivative function** denoted by $\frac{\mathrm{d}f}{\mathrm{d}\theta}$ or simply f'.
One of the most important tools for determining derivatives is the **chain rule**, which states that for two differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ we have

$$(f \circ g)'(\theta) = f'(g(\theta)) \cdot g'(\theta) \qquad \text{for all } \theta \in \mathbb{R}.$$

2.4.1 Measure Theory

In order to properly define integrals we will briefly recall the definition of the **Lebesgue-measure** on $\mathbb{R}$. A proper introduction on this topic can be found in every measure or real analysis text book, for example [7, 78, 80].

Borel σ-algebra and Lebesgue-Borel-measure. Let $\mathcal{E} := \{\, (a, b) \subseteq \mathbb{R} \mid a \leq b \,\}$ be the set of all open intervals, then the **Borel σ-algebra** $\sigma(\mathcal{E})$ is defined to be the smallest set of subsets in $\mathbb{R}$ with

(i) $\mathcal{E} \subseteq \sigma(\mathcal{E})$,

(ii) $A, B \in \sigma(\mathcal{E}) \quad \Rightarrow \quad A \setminus B \in \sigma(\mathcal{E})$,

(iii) $A_n \in \sigma(\mathcal{E})$ for $n \in \mathbb{N} \quad \Rightarrow \quad \bigcup_{n \in N} A_n \in \sigma(\mathcal{E})$.

The uniquely defined function $\mu\colon \sigma(\mathcal{E}) \to [0, \infty]$ with

(i) $\mu((a, b)) := b - a$,

(ii) $\mu(A \setminus B) := \mu(A) - \mu(B)$ for all $A, B \in \sigma(\mathcal{E})$ with $B \subseteq A$,

(iii) $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) := \sum_{n \in \mathbb{N}} \mu(A_n)$ for all pair-wise disjoint sequences $(A_n)_{n \in \mathbb{N}}$ in $\sigma(\mathcal{E})$,

is called **Lebesgue-Borel-measure**.

Null sets and Lebesgue-measure. The **null sets** for the Lebesgue-measure are defined as

$$\mathcal{N} := \{\, N \subseteq A \mid A \in \sigma(\mathcal{E}) \text{ with } \mu(A) = 0 \,\}\,.$$

Note that every singleton, and in consequence, every countable set is a null set. But there also exist examples of uncountable null sets. Importantly, a countable union of null sets is, again, a null set. As an extension we add all null sets to the σ-algebra and define the extended measure accordingly:

$$\mathcal{L} := \{\, A \cup N \mid A \in \sigma(\mathcal{E}) \text{ and } N \in \mathcal{N} \,\} \quad \text{and} \quad \lambda(A \cup N) := \mu(A) \quad \text{for all } A \in \sigma(\mathcal{E}) \text{ and } N \in \mathcal{N}.$$

This measure λ is called **Lebesgue-measure** and the sets in $\mathcal{L}$ are called **measurable**. Note that all sets $A \setminus N$ with $A \in \sigma(\mathcal{E})$ and $N \in \mathcal{N}$ are also contained in $\mathcal{L}$. From now on this will be the only measure to consider.

"Almost all". We often want to make statements S about the elements of a subset $A \subseteq \mathbb{R}$ that only hold up to some null set. Hence, a statement S holds for **almost all** $\theta \in A$ if there exists a null set $N \in \mathcal{N}$ such that S holds for all $\theta \in A \setminus N$. One of the most important statements is that a function $f := D \to \mathbb{R}$ is **almost everywhere** differentiable, which means that there exists a null set $N \in \mathcal{N}$ such that f is differentiable at all $\theta \in D \setminus N$. In particular, the function is allowed to have many "kinks" or even "jumps" in its function graph as long as their positions form a null set.
The following lemma is a further example for this and it plays an important role later on.

Lemma 2.3 (Differentiation rule for a minimum)**.** *For every element e of a finite set E let $T_e\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function that is differentiable almost everywhere and let $\ell(\theta) := \min_{e \in E} T_e(\theta)$ for all $\theta \geq 0$. It holds that ℓ is almost everywhere differentiable with*

$$\ell'(\theta) = \min_{e \in E'_\theta} T'_e(\theta) \tag{2.2}$$

for almost all $\theta \geq 0$ where $E'_\theta := \{\, e \in E \mid \ell(\theta) = T_e(\theta) \,\}$.

Proof. Let $\phi \geq 0$ such that all T_e, for all $e \in E$, are differentiable, which is almost everywhere. Since all functions T_e are continuous at ϕ we have for sufficiently small $\varepsilon > 0$ that $\ell(\phi + \xi) = \min_{e \in E'_\phi} T_e(\phi + \xi)$ for all $\xi \in [\phi, \phi + \varepsilon]$. It follows that

$$\lim_{\xi \searrow 0} \frac{\ell(\phi + \xi) - \ell(\phi)}{\xi} = \lim_{\xi \searrow 0} \min_{e \in E'_\phi} \frac{T_e(\phi + \xi) - \ell(\phi)}{\xi} = \min_{e \in E'_\phi} \lim_{\xi \searrow 0} \frac{T_e(\phi + \xi) - T_e(\phi)}{\xi} = \min_{e \in E'_\phi} T'_e(\phi).$$

Note that every point ϕ where all T_e are differentiable, but for which the left derivative of ℓ does not coincide with the right derivative of ℓ, is a proper crossing of at least two T_e functions. Therefore, these points are isolated and form a null set. Hence, we have $\ell'(\phi) = \min_{e \in E'_\phi} T'_e(\phi)$ for almost all $\phi \in \mathbb{R}_{\geq 0}$. $\square$

2.4.2 Integrals

Measurable functions. A function $f\colon D \to \mathbb{R}$ for $D \subseteq \mathbb{R}$ is called **(Lebesgue-)measurable** if the set $\{\, \theta \in D \mid f(\theta) < c \,\}$ is measurable for all $c \in \mathbb{R}$.

Simple functions. For a finite sequence of pair-wise disjoint measurable sets $A_1, A_2, \ldots, A_n$ and corresponding real numbers $(a_1, a_2, \ldots, a_n)$ we call the function $f(\theta) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(\theta)$ a **simple function**. Here, $\mathbb{1}_{A_k}$ denotes the **indicated function** defined by $\mathbb{1}_{A_i}(\theta) = 1$ if $\theta \in A_i$ and $\mathbb{1}_{A_i}(\theta) = 0$ otherwise. We define the **integral** of a simple function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$\int_{\mathbb{R}} f(\theta)\,\mathrm{d}\theta := \sum_{i=1}^{n} a_i \cdot \lambda(A_i).$$

Lebesgue-integral. For a measurable non-negative function $f\colon D \to \mathbb{R}$ we denote by $L(f)$ the set of all simple functions g with measurable $A_i \subseteq D$ and $g(\theta) \leq F(\theta)$ for all $\theta \in A_i$. With this we define the **integral** by

$$\int_D f(\theta)\,\mathrm{d}\theta := \sup_{g \in L(f)} \left\{ \int_{\mathbb{R}} g(\theta)\,\mathrm{d}\theta \right\}.$$

Note that the integral can be ∞. To obtain the integral for an arbitrary measurable function $f\colon D \to \mathbb{R}$ we split it into the positive and the negative part, i.e., we define $f^+(\theta) := \max\{\, f(\theta), 0 \,\}$ and $f^-(\theta) := \max\{\, -f(\theta), 0 \,\}$. We call f **quasi-integrable** if at least one of the functions f^+ or f^- has a finite integral and f is **integrable** if both, f^+ and f^-, have a finite integral. In either case we define

$$\int_D f(\theta)\,\mathrm{d}\theta := \int_D f^+(\theta)\,\mathrm{d}\theta - \int_D f^-(\theta)\,\mathrm{d}\theta.$$

Note that for quasi-integrable functions this value can be ∞ or $-\infty$.
We call a measurable function $f\colon D \to \mathbb{R}$ **locally integrable** if for every compact set $X \subseteq D$ we have that $f \cdot \mathbb{1}_X$ is integrable. In this case we write for every finite interval $[a, b]$

$$\int_a^b f(\theta)\,\mathrm{d}\theta := \int_D f(\theta) \cdot \mathbb{1}_{[a,b]}(\theta)\,\mathrm{d}\theta < \infty.$$

Since integrals are invariant on changes on null sets it does not matter for this notation whether the interval is closed or open, as long as the closed interval is contained in D.

Integral function. Given a locally integrable function $f\colon [0, \infty) \to \mathbb{R}$ we can define the **integral function** $F\colon [0, \infty) \to \mathbb{R}$ by

$$F(\theta) := \int_0^\theta f(\xi)\,\mathrm{d}\xi.$$

The **Lebesgue's differentiation theorem**, which can be seen as a generalization of the **fundamental theorem of calculus**, states the following.

Theorem 2.4 (Lebesgue's differentiation theorem).
Given a locally integrable function $f\colon [0, \infty) \to \mathbb{R}$, the integral function F is differentiable almost everywhere on $(0, \infty)$ and for its derivative it holds that for almost all $\theta \in (0, \infty)$ we have $F'(\theta) = f(\theta)$.

A proof can be found in most text books about Lebesgue integrals, for example in [78, 80].

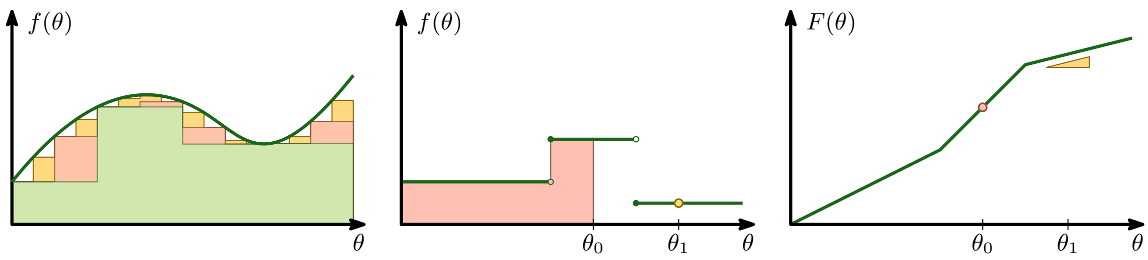


Figure 2.4: *On the left:* The integral of the function f equals the area beneath the function graph and is defined by the supremum of the integral of all simple functions with lower or equal function values. *In the middle:* The integral from 0 to θ_0 of the function f is given by the value of the integral function F at position θ_0 (*on the right*). In return the function F is almost everywhere differentiable with a derivative of $f(\theta_1)$ at position θ_1.

It follows immediately that if f is bounded by a bound L then F is L-Lipschitz continuous. Basic calculation rules known for the Riemann integral, such as the **integration by substitution**, also hold for Lebesgue integrals:

$$\int_{\ell(\phi_1)}^{\ell(\phi_2)} f(\theta)\,\mathrm{d}\theta = \int_{\phi_1}^{\phi_2} f(\ell(\xi)) \cdot \ell'(\xi)\,\mathrm{d}\xi.$$

Finally, we have the following theorem about monotone functions.

Theorem 2.5 (Lebesgue's theorem for the differentiability of monotone functions).
Every monotone function $F\colon D \to \mathbb{R}$ is almost everywhere differentiable on D.

A proof for this theorem can be found, for example, in the text book by Royden and Fitzpatrick [78].

2.5 Variational Inequalities and a Fixed Point Theorem

A powerful tool to prove the existence of dynamic equilibria are variational inequalities, which can be defined for finite dimensional vector spaces as well as for infinite dimensional function spaces. In this section we only present the essential definitions and results that we need throughout this thesis. For more details on this topic we refer to the book of Kinderlehrer and Stampaccia [49] and the more recent textbook of Facchinei and Pang [26].

2.5.1 Finite-dimensional Variational Inequalities

For a natural number n, we consider a set $X \subseteq \mathbb{R}^n$ and a function $\Gamma\colon X \to \mathbb{R}^n$. The **finite-dimensional variational inequality problem** $\mathrm{VI}(X,\Gamma)$ is the following.

$$\text{Find } x \in X \text{ such that} \quad (y-x)^\mathsf{T} \cdot \Gamma(x) \geq 0 \quad \text{for all } y \in X. \tag{VI}$$

From Brouwer's fixed point theorem, we obtain that the set of solutions $\mathrm{SOL}(X,\Gamma)$ is non-empty whenever X and Γ satisfy some conditions.

Theorem 2.6.
Let $X \subseteq \mathbb{R}^n$ be non-empty, compact and convex and let $\Gamma\colon X \to \mathbb{R}^n$ be a continuous mapping. Then $\mathrm{SOL}(X,\Gamma)$ is non-empty and compact.

For the proof we refer to [39].

Nonlinear complementarity. If X is a box, i.e., $X = \bigtimes_{i=1}^{n}[0, M_i]$ for some $M_i > 0$, $i = 1, \ldots, n$, it is easy to see that for a given solution $x^* \in \mathrm{SOL}(X, \Gamma)$ the **nonlinear complementarity property** holds for every $i = 1, 2, \ldots, n$ with $x_i^* < M_i$:

$$\Gamma_i(x^*) \geq 0 \qquad \text{and} \qquad x_i^* \cdot \Gamma_i(x^*) = 0. \tag{NCP}$$

The left inequality follows immediately since for $x_i^* < M_i$ we can choose y to be equal to x^* with the exception of $y_i := M_i$. Hence, by (VI) we have that $\Gamma_i(x^*)$ cannot be negative. The right equation follows with the same argumentation. For $x_i^* \in (0, M_i)$ we can choose y to be equal to x^* except for y_i, which can either be smaller or greater than x_i^*. Hence, (VI) implies that $\Gamma_i(x^*)$ must be 0 in this case.

2.5.2 Variational Inequalities in Infinite-dimensional Function Spaces.

We can transfer the concept of variational inequalities to infinite-dimensional Hilbert spaces as well, which we will consider next.

Scalar products. Given a vector space V over $\mathbb{R}$ a **scalar product** is a mapping $\langle \cdot, \cdot \rangle \colon V^2 \to \mathbb{R}$ that is linear in both components and satisfies

(i) **symmetry:** $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$.

(ii) **positive-definiteness:** $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.

Hilbert space L^2. For this section we consider two functions $f_1, f_2 \colon D \to \mathbb{R}$ to be equal if $f_1(\theta) = f_2(\theta)$ for almost all $\theta \in D$. In other words, two functions are equal if they only differ on a null set. More formally, that means that if we define f_1 and f_2 to be equivalent then we identify the functions with their respective equivalent classes.
This way we can define the L^2-space by

$$L^2(D) := \left\{ f \colon D \to \mathbb{R} \;\middle|\; \int_D (f(\theta))^2 \,\mathrm{d}\theta < \infty \right\}.$$

Together with the scalar product

$$\langle f, g \rangle = \int_D f(\xi) \cdot g(\xi) \,\mathrm{d}\xi$$

this forms an **Hilbert space**, which means that $L^2(D)$ together with the **induced norm**

$$\|f\| := \sqrt{\langle f, f \rangle}$$

is a **complete metric space**, i.e., all Cauchy sequences have a limit in $L^2(D)$. Given a natural number d we can consider the space of function-vectors

$$L^2(D)^d := \left\{ (f_i)_{i=1}^d \;\middle|\; f_i \in L^2(D) \text{ for all } i = 1, 2, \ldots, d \right\},$$

which again form a Hilbert space together with the scalar product

$$\langle f, g \rangle = \sum_{i=1}^{d} \int_D f_i(\xi) \cdot g_i(\xi) \,\mathrm{d}\xi.$$

We say a sequence $f^k \in L^2(D)^d$ **converges weakly** to $f \in L^2(D)^d$, if $\langle f^k, g\rangle \to \langle f, g\rangle$ for all $g \in L^2(D)^d$ and, for a given subset $X \subseteq L^2(D)^d$, we call a mapping $\mathcal{A}\colon X \to L^2(D)^d$ **weak-strong continuous** at $f \in X$, if for every $f^k \in X$ that converges weakly to f, we have that $\mathcal{A}(f^k)$ converges to $\mathcal{A}(f)$ with respect to the induced $L^2(D)^d$-norm. Note that strong convergence implies weak convergence but not the other way round. Hence, a weak-strong continuous mapping is always (strong-strong)-continuous but not vice versa. As an important example we show that the integration operator is weak-strong continuous.

Lemma 2.7. *The integration operator $f \mapsto F$ with $F_i(\theta) := \int_0^\theta f(\xi)\,\mathrm{d}\xi$ is a weak-strong continuous mapping $L^2(D)^d \to L^2(D)^d$ as long as $D = [0, H]$ is a compact interval.*

Proof. Consider a sequence of $(f^k)_{k\in\mathbb{N}}$ in $L^2(D)^d$ that converges weakly to f, i.e., $\langle f^k, g\rangle \to \langle f, g\rangle$ for all $g \in L^2(D)^d$. For every θ we consider the vector of functions g^θ with $g_i^\theta(\xi) = 1$ for $\xi = [0, \theta]$ and $g_i^\theta(\xi) = 0$ otherwise. We obtain that $\int_D g_i^\theta(\xi)\,\mathrm{d}\xi = \theta < \infty$, and therefore, $g^\theta \in L^2(D)^d$. Hence,

$$F_i^k(\theta) = \int_D f_i^k(\xi) \cdot g_i^\theta(\xi)\,\mathrm{d}\xi \to \int_D f_i(\xi) \cdot g_i^\theta(\xi)\,\mathrm{d}\xi = F_i(\theta).$$

This shows that F^k converges point-wise to F. In order to show the convergence in the L^2-norm, given an $\varepsilon > 0$, we consider a number $k_\theta \in \mathbb{N}$ for every $\theta \in D$ such that

$$\left|F_i^k(\theta) - F_i(\theta)\right| \leq \sqrt{\frac{\varepsilon}{2 \cdot H \cdot d}} \qquad \text{for all } i = 1, 2, \ldots, d \text{ and } k \geq k_\theta.$$

Since for increasing N the measure of $S := \{\, \theta \in D \mid k_\theta > N \,\}$ converges to 0 we can choose $N^* \in \mathbb{N}$ such that $\int_S \left|F_i^k(\xi) - F_i(\xi)\right|^2 \mathrm{d}\xi \leq \frac{\varepsilon}{2\cdot d}$ for all $N \geq N^*$. This way we obtain for all $k \geq N^*$ that

$$\begin{aligned}
\left\|F^k - F\right\|_{L^2}^2 &= \sum_{i=1}^d \int_D \left|F_i^k(\xi) - F_i(\xi)\right|^2 \mathrm{d}\xi \\
&\leq \sum_{i=1}^d \int_S \left|F_i^k(\xi) - F_i(\xi)\right|^2 \mathrm{d}\xi + \sum_{i=1}^d \int_{D\setminus S} \left|F_i^k(\xi) - F_i(\xi)\right|^2 \mathrm{d}\xi \\
&\leq \sum_{i=1}^d \frac{\varepsilon}{2 \cdot d} + \sum_{i=1}^d \int_{D\setminus S} \frac{\varepsilon}{2 \cdot H \cdot d}\,\mathrm{d}\theta \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, F^k converges to F in $L^2(D)^d$. □

Infinite-dimensional variational inequality. Given a subset $X \subseteq L^2(D)^d$ and a mapping $\mathcal{A} : X \to L^2(D)^d$, the variational inequality $\mathrm{VI}(X, \mathcal{A})$ is the following.

$$\text{Find } f \in X \text{ such that} \quad \langle \mathcal{A}(f), g - f\rangle \geq 0 \quad \text{for all } g \in X. \tag{infVI}$$

Brézis [10, Theorem 24] specifies conditions to guarantee the existence of a solution (see also [94]).

Theorem 2.8.

Let X be a non-empty, closed, convex and bounded subset of $L^2(D)^d$. Let $\mathcal{A} : X \to L^2(D)^d$ be a weak-strong continuous mapping. Then, the variational inequality $\mathrm{VI}(X, \mathcal{A})$ has a solution $f^* \in X$.

This result will be important for proving the existence of multi-commodity Nash flows over time in Section 5.1.4.

2.5.3 Kakutani's Fixed Point Theorem

Another important tool to show the existence of thin flows in later chapters is the fixed point theorem by Kakutani from 1941; see [48].

Theorem 2.9 (Kakutani's Fixed Point Theorem).
Let X be a compact, convex and non-empty subset of $\mathbb{R}^n$ and $\Gamma\colon X \to 2^X$, such that for every $x \in X$ the image $\Gamma(x)$ is non-empty and convex. Suppose the set $\{\, (x, y) \mid y \in \Gamma(x) \,\}$ is closed. Then there exists a fixed point $x^* \in X$ of Γ, i.e., $x^* \in \Gamma(x^*)$.

For the proof we refer to the original publication of Kakutani [48].

2.6 Linear and Integer Programming

Linear and integer programming are very useful tools to describe combinatorial optimization problems and there exist powerful solvers to compute solutions in reasonable time. In this thesis we will show how to utilize mixed integer programs (a combination of linear and integer programs) in order to algorithmically construct Nash flows over time, or more precisely, the underlying static thin flows.

Linear programming. A **linear program** in its **standard form** is given by

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

Here, $c \in \mathbb{R}^n$ is a **cost vector**, $A \in \mathbb{R}^{m \times n}$ is the **coefficient matrix** and $b \in \mathbb{R}^m$ is the **bound vector**. The conditions $Ax = b$ and $x \geq 0$ mean that the equality/inequality holds for every component of these vectors. We call a point $x \in \mathbb{R}^n$ **feasible** if it satisfies $Ax = b$ and $x \geq 0$. A feasible point is a **optimal solution** if it has the maximal **objective value** $c^\mathsf{T} x$ of all feasible points. These optimization problems can be solved very fast in practice, for example with the **simplex method**, and they can also be solved theoretically efficient via the **ellipsoid algorithm** or **interior point methods**, which will not be discussed in detail here. More important in regard of this thesis is the construction of a dual program, which is again a linear program. For a **primal** linear program in standard form as displayed above the **dual** program is given by

$$\begin{aligned} \min_{y \in \mathbb{R}^m} \quad & b^\mathsf{T} y \\ \text{s.t.} \quad & A^\mathsf{T} y \geq c. \end{aligned}$$

For every of the m conditions in the primal we have a variable y_i in the dual and vice versa. For every pair of feasible points (x, y) we have

$$b^\mathsf{T} y = x^\mathsf{T} A^\mathsf{T} y \geq x^\mathsf{T} c.$$

Hence, every feasible solution of the dual gives an upper bound on the optimal value of the primal and every feasible solution of the primal yields a lower bound on the optimal value of the dual. This property is called **weak duality**. For linear programs we even have **strong duality**, which means that the maximal objective value of the primal equals the minimal objective value of the dual.

Closely related to this is the so-called **complementary slackness**. This means that for a pair of optimal solutions for the primal and the dual (x^*, y^*) we have for all $i = 1, 2, \ldots, n$ that

$$x_i^* > 0 \;\Rightarrow\; (A^\mathsf{T} y^*)_i = c_i \qquad \text{and} \qquad (A^\mathsf{T} y^*)_i > c_i \;\Rightarrow\; x_i^* = 0.$$

These are also sufficient conditions, in other words, for a pair of feasible points (x, y) that satisfies the complementary slackness conditions, x is an optimal solution for the primal and y an optimal solution for the dual.

(Mixed) integer programming. If we consider a linear program as depicted above but additionally demand that all components of x are integers (i.e., $x \in \mathbb{Z}^n$) we obtain a so-called **integer (linear) program**. There are many different combinatorial problems that can be modeled by integer programs. Even though, there most likely does not exist any efficient algorithm, as solving an integer program is NP-complete, there are software-solvers that find a solution rather fast for most reasonable sized instances.
A variation of integer programs are **mixed integer programs** where only some of the variables have to be integers. Therefore, the standard form of these problems for a given index set $I \subseteq \{1, 2, \ldots, n\}$ is given by

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0, \\ & x_i \in \mathbb{Z} \quad \text{for } i \in I. \end{aligned}$$

Often it is enough to restrict the discrete variables only to be in the set $\{0, 1\}$. In this case, they are called **Boolean variables**. Unfortunately (mixed) integer programs with Boolean variables only are still NP-complete.
For more details on linear and (mixed) integer programming we refer to the textbook of Schrijver [85] and, for a more recent publication, to the book of Weismantel and Bertsimas [3].

2.7 Flows Over Time

As final topic of the preliminaries we want to give a brief introduction into flows over time without considering game-theoretical aspects. Instead we want to consider two types of optimization problems.

Feasible flows over time. Consider a network consisting of a directed graph $G = (V, E)$ with a source s and a sink t, such that t is reachable by s, as well as a capacity $\nu_e > 0$ and a transit time $\tau_e \geq 0$ for every arc $e \in E$. A family of locally integrable and bounded functions $f = (f_e)_{e \in E}$, where $f_e \colon [0, \infty) \to [0, \infty)$ denotes the inflow rate into arc e at time θ, is called **flow over time** if **flow conservation** is satisfied at every node $v \in V \setminus \{s, t\}$ for almost all $\theta \in [0, \infty)$:

$$\sum_{e \in \delta_v^-} f_e(\theta - \tau_e) = \sum_{e \in \delta_v^+} f_e(\theta).$$

Here, $f_e(\theta - \tau_e)$ denotes the outflow rate of arc e at time θ and since there cannot be any delay in this model, this equals the inflow rate at time $\theta - \tau_e$. Note that for $\theta < \tau_e$ we define this value as 0.

The **value function** of a flow over time is given by the **excess** at sink t for each point in time:

$$|f|(\theta) := \sum_{e \in \delta_t^-} \int_0^{\theta - \tau_e} f_e(\xi)\,\mathrm{d}\xi - \sum_{e \in \delta_t^+} \int_0^{\theta} f_e(\xi)\,\mathrm{d}\xi.$$

We call a flow over time f **feasible** if $f_e(\theta) \leq \nu_e$ holds for all $e \in E$ and almost all $\theta \in [0, \infty)$.

Optimal Flows Over Time Without giving too much details we want to present two typical optimization problems for flows over time.

A typical optimization problem is the **maximum flow over time problem** that asks for a given time horizon $H \geq 0$ to determine a feasible flow over time f with maximal $|f|(H)$. This problem can be solved with the help of so-called **temporally repeated flows**, which are feasible flows over time corresponding to a path decomposition of a feasible static s-t-flow. The key idea is to send a constant flow rate of x_P into each simple path $P \in \mathcal{P}$ during the interval $[0, H - \sum_{e \in P} \tau_e]$. This leads to an objective value of

$$H \cdot |x| - \sum_{e \in E} \tau_e \cdot x_e.$$

It turns out that a temporally repeated flow with an underlying feasible static flow that maximizes this value is indeed a maximum flow over time and it can be computed efficiently by an algorithm of Ford and Fulkerson. For details on this we refer to the survey of Skutella [89].

The **quickest flow problem** for a given flow volume $M \geq 0$ is to find a flow over time that sends M units of flow as fast as possible from the source to the sink. In other words, the goal is to minimize the arrival time of the particle that arrives last. As mentioned in the introduction this can be solved by combining the algorithm of Ford and Fulkerson for the maximum flow over time problem with a binary search framework (or faster with parametric search); see [29, 11].

The temporally repeated flows for the maximum flow over time problem are constructed specifically for a given time horizon H, and therefore, we cannot expect that the value of a maximum flow over time is also optimal for earlier points in time $\theta < H$. Surprisingly, it is possible to construct a feasible flow over time that is maximal for all times simultaneously, and in addition, solves the quickest flow problem for all flow volumes at the same time. These flows over time are called **earliest arrival flows** and they can be constructed by considering **generalized temporally repeated flows**; see Figure 2.5 for an example. By considering the transit times as costs we can apply the successive shortest path algorithm (see optimal static flows on page 12) in order to obtain a **generalized path decomposition** $(x_P)_{P \in \mathcal{P}^*}$. Note that in this decomposition paths are allowed to use an arc in the inverse direction. The earliest arrival flow is then obtained by sending a constant flow rate of x_P into each of these augmenting paths. This means that we might send a negative flow rate in the inverse direction back in time. But it turns out that whenever this happens, this negative flow cancels out with a positive flow that is sent in the forward direction. For the proof that such a flow is always well-defined and that it is indeed an earliest arrival flow we refer to [89].

2.8 Notation

In order to make this thesis as comprehensible as possible we try to keep the notation intuitive and consistent. Throughout this thesis f is used for a flow over time consisting of functions representing flow rates that are (locally) integrable but in general not differentiable. For the integral functions, which are almost everywhere differentiable, we use the corresponding capital letter F. Graphs

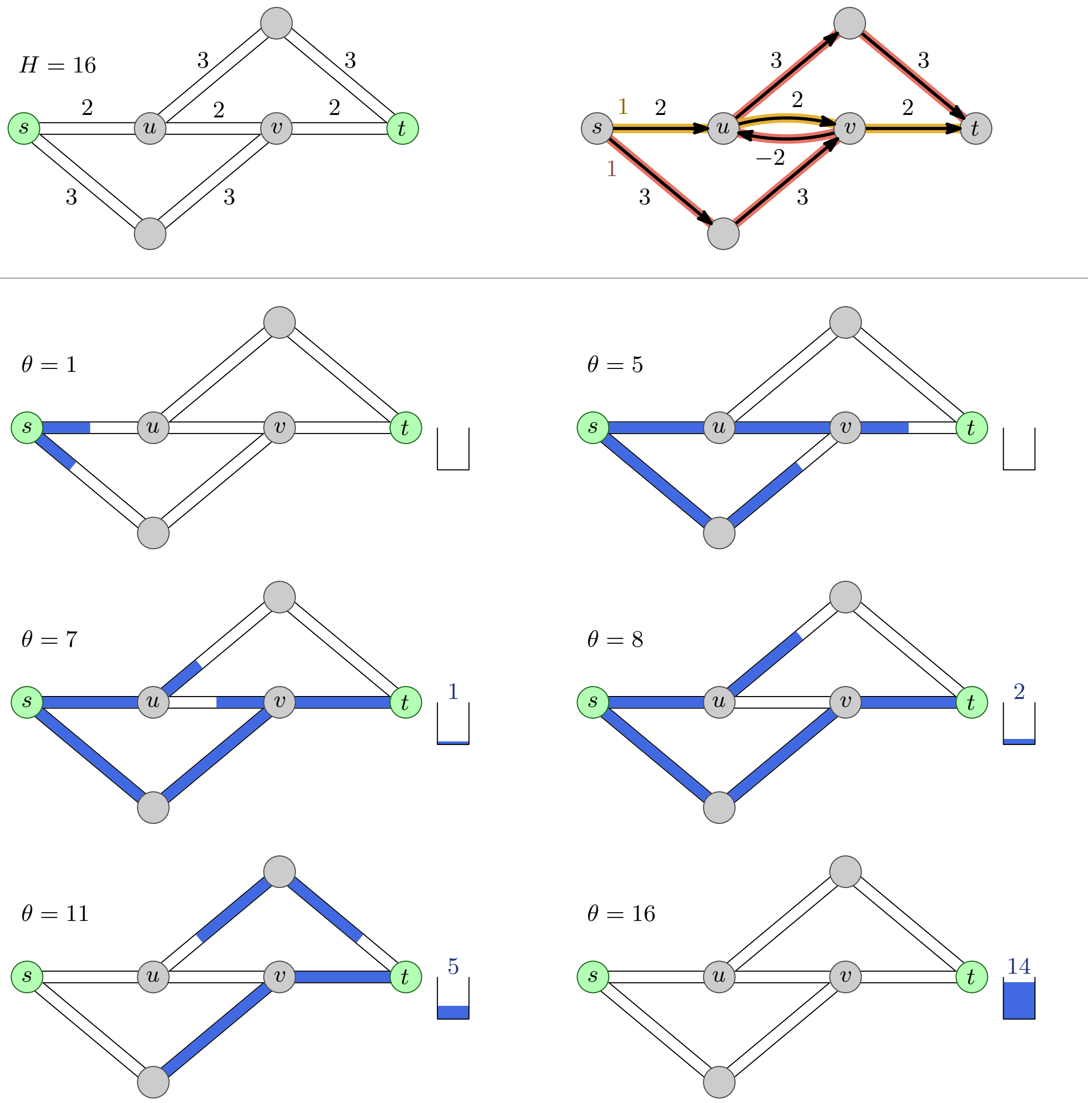


Figure 2.5: An earliest arrival flow at different snapshots in time in the network depicted *in the upper left*. The labels denote the transit times and we consider unit capacities as well as a time horizon of $H = 16$. All arcs are orientated from left to right. The successive shortest path algorithm (*top right*) picks the direct route (s, u, v, t) first and sends one unit of flow there. Next, it chooses the long augmenting path that includes the backward arc vu. In the earliest arrival flow this corresponds to a negative flow over time that is sent from v to u backwards in time from $\theta = 8$ to $\theta = 6$. *In the bottom left,* we observe that at time $\theta = 7$ this negative flow cancels out with the positive forward traveling flow between u and v.

are denoted by $G = (V, E)$ with source s and sink t and the letters u, v, w are reserved for nodes, whereas e is used for arcs, and only very rarely for the Euler constant. The non-negative real numbers are normally denoted by $[0, \infty)$ (or $[0, \infty]$ if ∞ is included). This notation is used whenever we denote flow rates or points in time (denoted by θ or ϑ) and only if we consider the flow as set of all particles (denoted by ϕ or φ) we write $\mathbb{R}_{\geq 0}$. The set of commodities is, in general, given by J and the individual commodities are denoted by either j or i. Finally, x and ℓ are reserved for the underlying static flow and the earliest arrival times and their derivatives are denoted by x' and ℓ'.

3 The Base Model

In this chapter we consider dynamic equilibria in the most basic flow over time model with deterministic queuing. Even though the flow dynamics were already mentioned by Vickrey in 1969 [96], Koch and Skutella were the first to consider Nash flows over time within this model [54] (for the full version see [55]). Cominetti, Correa, Larré, Olver and others extended the theory during the following years [17, 18, 19, 20].

At the beginning of this chapter we give an intuitive as well as a formal definition of the flow over time model with deterministic queuing in Sections 3.1 and 3.2. After defining Nash flows over time and thin flows with resetting, we present the main results of the aforementioned publications in Sections 3.3 and 3.4. Finally, we give a short overview of the computation and the most recent results of dynamic equilibria in this base model, namely the long-term behavior and the price of anarchy, in Section 3.6.

It is worth noting that for the sake of consistency to later chapters the notation of the presented model slightly differs from the notation in literature. On the one hand, we always imagine the queues to be at the head of the arcs and, on the other hand, we consistently differentiate between a particle ϕ and the time it enters the network $\ell(\phi)$. This adapted notation is completely equivalent to the original formulation from literature but both of these modifications will be important for extended models in later chapters.

3.1 Modeling Road Networks

In order to motivate flows over time with deterministic queuing and to give some intuition on the flow dynamics we present a model of a street segment and show how it translates to an arc in a flow over time network.

Road model. Consider the following simplified discrete model of a road network. Each street segment is equipped with a length ℓ, a width w (number of lanes) and a speed limit v; see left side of Figure 3.1. Each street segment starts and ends at a street node, for example an intersection, and the segment is one-directional.

Traffic is denoted by two functions $f^+, f^- : \mathbb{N}_0 \to \mathbb{N}_0$, where $f^+(\theta)$ describes the number of vehicles entering and $f^-(\theta)$ the number of cars leaving the segment at time step θ. After entering a street it takes ℓ/v time to traverse the segment unless there is some congestion. The width induces an upper limit on the number of vehicles that can traverse the street per time unit, which we call capacity. If more cars enter the street than the capacity allows the traffic users have to reduce the velocity, which leads to congestion. We consider the simplest model of traffic congestion.

The number of vehicles that are allowed to leave a road segment is restricted by the capacity and if more cars want to leave they have to line up in a queue in front of the exit. In summary, a vehicle entering the segment first moves with maximal speed of v until it hits the end of the traffic jam. There it lines up and waits until it leaves the street in order to enter a successive line segment.

Note that overtaking is not allowed within one segment and streets can never become full for now.

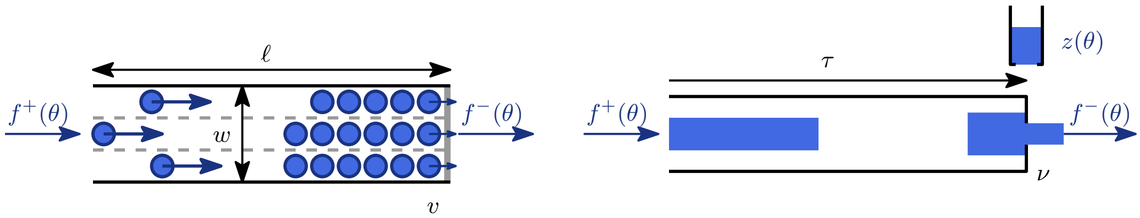


Figure 3.1: *On the left:* A road segment of length ℓ and width w with speed limit v. Atomic vehicles enter over time, traverse the arc at speed v before lining up at the tail of the queue. Only the particles at the front can leave the arc. *On the right:* An arc with transit time τ and capacity ν as we consider it in the base model. A flow over time is defined by the inflow rate $f^+(\theta)$ and the outflow rate $f^-(\theta)$ for every point in time $\theta \in [0, \infty)$. Additionally, we consider the queue size $z(\theta)$ at time θ as only a rate of ν can leave the arc at any point in time.

Arc model. In the flow over time model the traffic is described by a continuous flow over time in a directed graph; see right side of Figure 3.1. Each street segment corresponds to a directed arc between two nodes. Every arc is hereby equipped with a free flow transit time τ corresponding to ℓ/v and a capacity ν corresponding to w.
Incoming and outgoing traffic flow is described by two functions $f^+, f^- : [0, \infty) \to [0, \infty)$, which denote the inflow and outflow rate at each point in time.
If the inflow rate exceeds the capacity a point queue is built up right before the head of the arc, where the particles line up. Note that the flow on every arc has to follow the FIFO principle, hence, no particle can overtake any other particle.

3.2 Deterministic Queuing Model

In this section we define the dynamics of flows over time in the deterministic queuing model in a mathematical precise way. This section is heavily based on the formulation introduced by Koch and Skutella [54] that was later improved by Cominetti, Correa and Larré [17].

Networks. A *flow over time network* in the base model consists of a directed graph $G = (V, E)$ with a source s and a sink t, such that each node is reachable by s. Furthermore, we have a constant network inflow rate at s of $r > 0$ and every arc e is equipped with a transit time $\tau_e \geq 0$ and a capacity $\nu_e > 0$. We assume that the sum of transit times in every directed cycle is strictly positive.

Flows over time. A flow over time is specified by a family of functions $f = (f_e^+, f_e^-)_{e \in E}$, where $f_e^+, f_e^- : [0, \infty) \to [0, \infty)$ are locally integrable and bounded functions for every arc e. Hereby, the function f_e^+ describes the **inflow rate** and f_e^- the **outflow rate** of arc e for every given point in time $\theta \in [0, \infty)$. The **cumulative in- and outflow** of an arc e is the total amount of flow that has entered or left e up to some point in time θ and is defined by

$$F_e^+(\theta) := \int_0^\theta f_e^+(\xi)\,\mathrm{d}\xi \quad \text{and} \quad F_e^-(\theta) := \int_0^\theta f_e^-(\xi)\,\mathrm{d}\xi.$$

Due to technical reasons we define $f_e^+(\theta) = f_e^-(\theta) = F_e^+(\theta) = F_e^-(\theta) = 0$ for all $\theta < 0$.

Flow conservation. We say $f = (f_e^+, f_e^-)_{e \in E}$ is a **flow over time** if it **conserves flow on all arcs** e:

$$F_e^-(\theta + \tau_e) \leq F_e^+(\theta) \qquad \text{for all } \theta \in [0, \infty), \tag{3.1}$$

and if it **conserves flow at every node** $v \in V\setminus\{t\}$, which means that the following equation holds for almost all $\theta \in [0,\infty)$:

$$\sum_{e\in\delta_v^+} f_e^+(\theta) - \sum_{e\in\delta_v^-} f_e^-(\theta) = \begin{cases} 0 & \text{if } v \in V\setminus\{t\}, \\ r & \text{if } v = s. \end{cases} \tag{3.2}$$

Particles entering arc e at time θ need τ_e time to traverse the arc, and therefore, they reach the head of e earliest at time $\theta + \tau_e$. Hence, (3.1) ensures that the amount of flow that leaves e cannot exceed the amount of flow that has entered and traversed the arc. In other words, flow is not created within arcs. In addition, (3.2) ensures that the network does not leak at intermediate nodes and that we have a constant network inflow rate of r at s. Note that flow can be stored within arcs but never on nodes.

Queues. For every arc e there is a bottleneck given by its capacity ν_e right before the head of the arc. If more flow wants to leave e than the capacity allows, a queue builds up at the exit of the arc. The amount of flow in the queue at time θ is given by

$$z_e(\theta) := F_e^+(\theta - \tau_e) - F_e^-(\theta).$$

The queue does not have any physical dimension in the network and is therefore called **point queue**.

Feasibility. In the base model we say a flow over time f is **feasible** if the queue operates at capacity, that means we have

$$f_e^-(\theta) = \begin{cases} \nu_e & \text{if } z_e(\theta) > 0, \\ \min\{\, f_e^+(\theta - \tau_e), \nu_e \,\} & \text{else,} \end{cases} \qquad \text{for almost all } \theta \in [0,\infty). \tag{3.3}$$

Note that this equation together with the definition of z_e immediately implies (3.1), which shows that a family of locally integrable and bounded functions $f = (f_e^+, f_e^-)_{e\in E}$ is already a feasible flow over time if only (3.2) and (3.3) are satisfied.

Waiting times. Given a feasible flow over time f, a particle entering an arc e at time θ first traverses the arc in τ_e time units and then queues up and has to wait in line. Hence, the **waiting time** $q_e\colon [0,\infty) \to [0,\infty)$ is given by

$$q_e(\theta) := \frac{z_e(\theta + \tau_e)}{\nu_e}.$$

Note that $q_e(\theta)$ denotes the waiting time of particles that enter the *arc* at time θ, and therefore, they enter the queue only at time $\theta + \tau_e$. Hence, the actual waiting period of those particles is given by $[\theta + \tau_e, \theta + \tau_e + q_e(\theta)]$.

Exit times. The **exit time** for arc e is the function $T_e\colon [0,\infty) \to [0,\infty)$ that maps the entrance time θ to the time a particle leaves the arc

$$T_e(\theta) := \theta + \tau_e + q_e(\theta).$$

Suppose a flow particle enters e at time θ, then the amount of flow which has entered e before θ is exactly the amount of flow that has left e before time $T_e(\theta)$, when the particle leaves the arc. This and other technical properties are collected in the following lemma.

Lemma 3.1. *For a feasible flow over time f it holds for all $e \in E$, $v \in V$ and $\theta \in [0, \infty)$ that:*

(i) $q_e(\theta) > 0 \quad \Leftrightarrow \quad z_e(\theta + \tau_e) > 0.$

(ii) $z_e(\theta + \tau_e + \xi) > 0$ *for all* $\xi \in [0, q_e(\theta))$.

(iii) $F_e^+(\theta) = F_e^-(T_e(\theta))$.

(iv) For $\theta_1 < \theta_2$ *with* $F_e^+(\theta_2) - F_e^+(\theta_1) = 0$ *and* $z_e(\theta_2 + \tau_e) > 0$ *we have* $T_e(\theta_1) = T_e(\theta_2)$.

(v) The functions T_e *are monotonically increasing.*

(vi) The functions F_e^+, F_e^-, z_e, q_e *and* T_e *are almost everywhere differentiable.*

(vii) For almost all $\theta \in [0, \infty)$ *we have*

$$q_e'(\theta) = \begin{cases} \frac{f_e^+(\theta)}{\nu_e} - 1 & \text{if } q_e(\theta) > 0, \\ \max\left\{ \frac{f_e^+(\theta)}{\nu_e} - 1, 0 \right\} & \text{else.} \end{cases}$$

Most of the statements follow immediately from the definitions and some involve some minor calculations. For (vi) we use Lebesgue's differentiation theorem (Theorem 2.4 on page 18). As the proof does not give any interesting further insights we moved it to the appendix on page 46.

3.3 Nash Flows Over Time

In this section we define a dynamic equilibrium, called **Nash flow over time**, and state the most important characterizations. In a dynamic equilibrium each particle of the flow can be seen as a separate player that tries to minimize its travel time from the source to the sink.

Earliest arrival times. The **source arrival time function** maps each particle $\phi \in \mathbb{R}_{\geq 0}$ to the time it arrives at s and is therefore given by $T_s(\phi) := \frac{\phi}{r}$. For an s-v path $P = (e_1, e_2, \dots, e_k)$ the **arrival time function** $T_P : \mathbb{R}_{\geq 0} \to [0, \infty)$ maps the particle ϕ to the time at which ϕ arrives at v if it traverses the path P, hence, we define

$$T_P(\phi) := T_{e_k} \circ T_{e_{k-1}} \circ \dots \circ T_{e_1} \circ T_s(\phi).$$

The **earliest arrival time function** $\ell_v : \mathbb{R}_{\geq 0} \to [0, \infty)$ of node $v \in V$ maps each particle ϕ to the earliest time $\ell_v(\phi)$ it can possibly reach node v. We have

$$\ell_v(\phi) := \min_{P \in \mathcal{P}_v} T_P(\phi),$$

where $\mathcal{P}_v$ is the set of all paths from s to v. Since all directed cycles in G have positive travel times, the earliest arrival times of every particle ϕ are characterized by the following equations:

$$\begin{aligned} \ell_s(\phi) &= T_s(\phi) = \frac{\phi}{r}, \\ \ell_v(\phi) &= \min_{e = uv \in E} T_e(\ell_u(\phi)) \quad \text{for } v \in V \setminus \{ s \}. \end{aligned} \tag{3.4}$$

Clearly, for given waiting time functions, the earliest arrival times can be computed efficiently by, for example, Dijkstra's algorithm. Furthermore, note that for every $v \in V$ the function ℓ_v

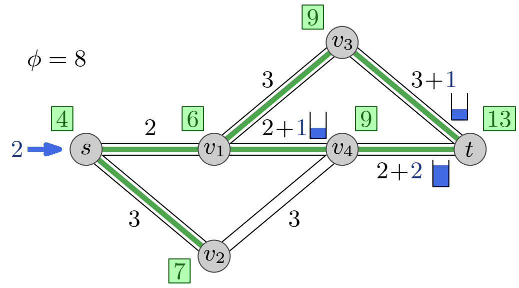


Figure 3.2: The current shortest paths for particle $\phi = 8$. Each arc is labeled with the transit time and the framed labels near the nodes display the earliest arrival times for this particle. If there is a positive queue at the respective time the corresponding waiting time is shown in blue. With a network inflow rate of 2, particle $\phi = 8$ enters the network at time $\ell_s(8) = 4$. As there are no queues on the first arcs the particle could arrive earliest at time $\ell_{v_1}(8) = 6$ at node v_1 and $\ell_{v_2}(8) = 7$ at node v_2. As there is a queue on arc v_1v_4 at time 6 the particle experiences a waiting time of $q_{v_1v_4}(6) = 1$ leading to an earliest arrival time at v_4 of $\ell_{v_4}(8) = 9$. In this particular instance all arcs except for v_2v_4 are active. Especially, the paths (s, v_1, v_4, t) and (s, v_1, v_3, t) are both current shortest paths.

is monotonically increasing since it is a minimum of monotonically increasing functions T_e (see Lemma 3.1 (v)). Hence, by Theorem 2.5 every earliest arrival time function ℓ_v is almost everywhere differentiable. We denote the derivative by ℓ'_v.

Current shortest paths networks and active arcs. In an equilibrium every particle wants to get to the sink as fast as possible and will therefore use a shortest path. For a fixed particle ϕ we call an arc $e = uv$ **active** for ϕ if $\ell_v(\phi) = T_e(\ell_u(\phi))$ holds. These are exactly the arcs that can be used in order to be as fast as possible. We denote the set of all active arcs for particle ϕ by

$$E'_\phi := \{\, e = uv \in E \mid \ell_v(\phi) = T_e(\ell_u(\phi)) \,\}$$

and the subgraph $G'_\phi = (V, E'_\phi)$ is called the **current shortest paths network**. An example of the earliest arrival times and a current shortest paths network is depicted in Figure 3.2.

Resetting arcs. It will be important to specify the arcs at which a particle would experience a waiting time when traveling along a shortest path. Hence, we call the set

$$E^*_\phi := \{\, e = uv \in E \mid q_e(\ell_u(\phi)) > 0 \,\}$$

the **resetting arcs** of particle ϕ.

Dynamic equilibria. Since every particle wants to arrive at the sink t as early as possible, it should only use current shortest paths, which leads to the following definition.

Definition 3.2 (Nash flow over time).
A feasible flow over time f is a **Nash flow over time**, also called **dynamic equilibrium**, if the following **Nash flow condition** holds:

$$f_e^+(\theta) > 0 \;\Rightarrow\; \theta \in \ell_u(\Phi_e) \qquad \text{for all arcs } e = uv \in E \text{ and almost all } \theta \in [0, \infty), \tag{N}$$

where $\Phi_e := \{\, \phi \in \mathbb{R}_{\geq 0} \mid e \in E'_\phi \,\}$ is the set of flow particles for which arc e is active.

Figuratively speaking, this condition means that a Nash flow over time uses only active arcs, and therefore only shortest paths to the sink t. More precisely, particle ϕ reaches t at time $\ell_t(\phi)$ by using active arcs only and $\ell_t(\phi)$ is the earliest time ϕ can possibly reach t under the assumption that the routes of all previous particles $\varphi < \phi$ are fixed. Since this is true for all particles, a Nash flow over time is indeed a Nash equilibrium in a game with a continuum of players.

Lemma 3.3 (cf. Theorem 1 in [17]). *Let f be a feasible flow over time, $\Phi_e := \{\, \phi \in \mathbb{R}_{\geq 0} \mid e \in E'_\phi \,\}$ the set of all particles for which e is active and $\Phi_e^c := \mathbb{R}_{\geq 0} \setminus \Phi_e$ its complement. Then the following statements are equivalent:*

(i) f is a Nash flow over time.

(ii) For each arc $e = uv$, it holds that $f_e^+(\theta) = 0$ for almost all $\theta \in \ell_u(\Phi_e^c)$.

(iii) $F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi))$ holds for all arcs $e = uv$ and all particles ϕ.

(iv) For every arc $e = uv$ and almost all $\phi \in \Phi_e^c$ we have $f_e^+(\ell_u(\phi)) \cdot \ell'_u(\phi) = 0$.

(v) For all ϕ and every arc $e = uv$ we have: If $F_e^+(\ell_u(\phi) - \varepsilon) < F_e^+(\ell_u(\phi))$ for all $\varepsilon > 0$ then $e \in E'_\phi$.

Especially the equations in (iii) are essential for the following considerations and the intuitive idea why they hold in a Nash flow over time is the following. Either arc e is active, then the equation follows from Lemma 3.1 (iii) or e is not active, but then the Nash condition states that there was no inflow between the last point in time θ when this arc was active and $\ell_u(\phi)$. Hence, we have $F_e^+(\ell_u(\phi)) = F_e^+(\theta) = F_e^-(T_e(\theta)) \leq F_e^-(\ell_v(\phi))$, which together with the flow conservation on arcs (3.1) shows that $F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi))$. The equivalences of (i) to (iv) were first shown in [17, Theorem 1] and a detailed version of the proof can be found in the appendix on page 46.
As a further result we observe that queues can only occur on active arcs in a Nash flow over time and the earliest arrival times characterize which arcs are active and which are resetting.

Lemma 3.4 (cf. Proposition 2 in [17]). *Given a Nash flow over time the following holds for all particles ϕ:*

(i) $E_\phi^* \subseteq E'_\phi$.

(ii) $E'_\phi = \{\, e = uv \mid \ell_v(\phi) \geq \ell_u(\phi) + \tau_e \,\}$.

(iii) $E_\phi^* = \{\, e = uv \mid \ell_v(\phi) > \ell_u(\phi) + \tau_e \,\}$.

The main idea of (i) is that for a resetting arc we have by definition that arc $e = uv$ has a positive waiting time at $\ell_u(\phi)$. If e would not be active for ϕ then it would also have not been active for the last particles in the queue of e. But this is a contradiction to the Nash condition. The detailed proof, first given in [17, Proposition 2], can be found in the appendix on page 49.

Underlying static flows. Lemma 3.3 (iii) motivates to consider the **underlying static flow** for every particle ϕ, which is defined by

$$x_e(\phi) := F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi)).$$

For a fixed particle ϕ this is indeed a static s-t-flow since the integral of (3.2) over $[0, \ell_v(\phi)]$ yields

$$\sum_{e \in \delta_v^+} x_e(\phi) - \sum_{e \in \delta_v^-} x_e(\phi) = \begin{cases} 0 & \text{if } v \in V \setminus \{\, s, t \,\}, \\ \phi & \text{if } v = s. \end{cases} \tag{3.5}$$

By taking the derivative of both sides, which exists for almost all ϕ according to Theorem 2.4, we obtain

$$\sum_{e\in\delta_v^+} x_e'(\phi) - \sum_{e\in\delta_v^-} x_e'(\phi) = \begin{cases} 0 & \text{if } v \in V \backslash \{\, s, t \,\}, \\ 1 & \text{if } v = s. \end{cases} \tag{3.6}$$

Note that it is possible to reconstruct the in- and outflow function of every arc $e = uv$ by these derivatives $x_e'(\phi)$ together with the derivatives of the earliest arrival times ℓ_u' and ℓ_v'. This becomes clear when we apply the chain rule to determine the derivatives of $x_e = F_e^+ \circ \ell_u = F_e^- \circ \ell_v$. We have

$$x_e'(\phi) = f_e^+(\ell_u(\phi)) \cdot \ell_u'(\phi) = f_e^-(\ell_v(\phi)) \cdot \ell_v'(\phi). \tag{3.7}$$

Consequently, a Nash flow over time is completely characterized by these derivatives, which have a very specific structure that is analyzed in the next section.

3.4 Thin Flows with Resetting

In the considered flow over time game every particle does not only choose one s-t-path but it can also split up even further and each part can take a different path from s to t. Hence, a strategy of a particle ϕ is, in fact, a convex combination of such paths, or in other words, a strategy is given by a static s-t-flow of value 1.

It turns out that for Nash flows over time the strategies of almost all particles have a specific structure, called **thin flows with resetting**. These are static s-t-flows on the respective current shortest paths network together with real-valued node labels. A preliminary form of these flows was introduced by Koch and Skutella in [54] and Cominetti et al. sharpened the definition in [18] and called them *normalized thin flows with resetting*. Throughout this thesis we will omit the “normalized” even though we still use a definition equivalent to the formulation in [18].

Thin flows with resetting. Let $G' = (V, E')$ be an acyclic subgraph of G such that every node is reachable by s. Note that not every node needs to be able to reach the sink. Additionally, we consider a subset of arcs $E^* \subseteq E'$, called **resetting arcs**.

Definition 3.5 (Thin flow with resetting).
A static s-t flow $(x_e')_{e\in E}$ in G' of value 1 together with a node labeling $(\ell_v')_{v\in V}$ is called **thin flow with resetting** on E^* if:

$$\ell_s' = \frac{1}{r} \tag{TF1}$$

$$\ell_v' = \min_{e=uv\in E'} \rho_e(\ell_u', x_e') \qquad \text{for all } v \in V \setminus \{\, s \,\}, \tag{TF2}$$

$$\ell_v' = \rho_e(\ell_u', x_e') \qquad \text{for all } e = uv \in E' \text{ with } x_e' > 0, \tag{TF3}$$

$$\text{where} \qquad \rho_e(\ell_u', x_e') := \begin{cases} \frac{x_e'}{\nu_e} & \text{if } e = uv \in E^*, \\ \max\left\{ \ell_u', \frac{x_e'}{\nu_e} \right\} & \text{if } e = uv \in E' \backslash E^*. \end{cases}$$

The intuitive idea behind these equations is that $\frac{x_e'}{\nu_e}$ describes the **stress value** of an arc e if a fraction of x_e' of the particles decides to use this arc. The stress value of an s-v-path is then determined by the highest stress value of its arcs (i.e., by the bottleneck arc along the path), as long as there are no resetting arcs; see $\rho_e(l_u', x_e')$ if $e \notin E^*$. At every resetting arc along the path the values of all previous arcs are dismissed and the stress value of the path is reset; see $\rho_e(l_u', x_e')$ if $e \in E^*$. This is logical

since a high stress value of preceding arcs can be compensated by decreasing the queue, as long as there is a positive queue.
Finally, the value ℓ'_v is the minimal stress value of all paths from s to v; see (TF2). For high stress values following particles need more time to reach node v, hence, these stress values coincide exactly with the slope of the earliest arrival time functions.
If $\ell'_v < \rho_e(\ell'_u, x'_e)$, this means that e leaves the current shortest paths network, and therefore, it cannot be used by following particles, i.e., $x'_e = 0$. In other words, an interval of particles in a Nash flow over time can only use arcs that lie on a path with minimal stress value; see (TF3).
Furthermore, $\ell'_u < \frac{x'_e}{\nu_e}$ means that arc e is a bottleneck, and therefore, the queue will grow. Whenever we have $\ell'_u > \frac{x'_e}{\nu_e}$ the arc e has a smaller stress value than the preceding arcs along the s-v-path. Hence, the queue will decrease if e is resetting, or stay empty otherwise. For $\ell'_u = \frac{x'_e}{\nu_e}$ the arc has the exact same stress value as the arcs before, so the queue will stay constant.
Examples of thin flows with resetting are displayed in the middle of Figure 3.3 on page 40.
As the first central structural result on Nash flows over time, the next theorem states that the derivatives of a Nash flow over time form, almost everywhere, a thin flow in the current shortest paths network with resetting on the set of arcs with positive queues.

Theorem 3.6 (cf. Theorem 9 in [54]).
For a Nash flow over time $f = (f_e^+, f_e^-)_{e\in E}$ the derivatives $(x'_e(\phi))_{e\in E'_\phi}$ together with $(\ell'_v(\phi))_{v\in V}$ form a thin flow with resetting on E^*_ϕ in the current shortest paths network $G'_\phi = (V, E'_\phi)$ for almost all $\phi \in \mathbb{R}_{\geq 0}$.

This theorem was first proven by Koch and Skutella for the preliminary form of thin flows in [54, Theorem 9]. For the improved formulation of thin flows a proof was given in [17, Theorem 2]. The proof we present in the following is similar to the one in [17], but we were able to keep the case distinctions to a minimum.

Proof. By Lemma 3.4 (i) we have that $E^*_\phi \subseteq E'_\phi$, and furthermore, G_ϕ is acyclic since the total transit time of every directed cycle is strictly positive. In other words, the preconditions for a thin flow are satisfied.
We have to show that Equations (TF1) to (TF3) hold for almost all particles. For this let ϕ be a particle such that for all $e = uv$ the derivatives of x_e, ℓ_v, and $T_e \circ \ell_u$ exist and $x'_e(\phi) = f_e^+(\ell_u(\phi)) \cdot \ell'_u(\phi) = f_e^-(\ell_v(\phi)) \cdot \ell'_v(\phi)$. This is given for almost every particle $\phi \in \mathbb{R}_{\geq 0}$.

(TF1) follows directly from (3.4).

(TF2) can be shown by taking the derivative of $T_e(\theta)$. By Lemma 3.1 (vii) we have

$$T'_e(\theta) = 1 + q'_e(\theta) = \begin{cases} \frac{f_e^+(\theta)}{\nu_e} & \text{if } q_e(\theta) > 0, \\ \max\left\{\frac{f_e^+(\theta)}{\nu_e}, 1\right\} & \text{else.} \end{cases}$$

We obtain

$$\frac{\mathrm{d}}{\mathrm{d}\phi} T_e(\ell_u(\phi)) = T'_e(\ell_u(\phi)) \cdot \ell'_u(\phi) = \left\{ \begin{array}{ll} \frac{x'_e(\phi)}{\nu_e} & \text{if } q_e(\ell_u(\phi)) > 0, \\ \max\left\{\frac{x'_e(\phi)}{\nu_e}, \ell'_u\right\} & \text{else,} \end{array} \right\} = \rho_e(\ell'_u(\phi), x'_e(\phi)).$$

Applying the differentiation rule for a minimum (Lemma 2.3 on page 17) on the equations (3.4) yields (TF2).

(TF3) is satisfied because for $x'_e(\phi) = f_e^-(\ell_v(\phi)) \cdot \ell'_v(\phi) > 0$ we have

$$\ell'_v(\phi) = \frac{x'_e(\phi)}{f_e^-(\ell_v(\phi))} = \begin{cases} \dfrac{x'_e(\phi)}{\min\{\, f_e^+(\ell_u(\phi)), \nu_e \,\}} & \text{if } z_e(\ell_u(\phi)+\tau_e) = 0, \\ \dfrac{x'_e(\phi)}{\nu_e} & \text{else,} \end{cases}$$
$$= \left\{ \begin{array}{ll} \max\left\{\, \ell'_u, \frac{x'_e(\phi)}{\nu_e} \,\right\} & \text{if } e \in E'_\phi \backslash E^*_\phi, \\ \frac{x'_e(\phi)}{\nu_e} & \text{if } e \in E^*_\phi, \end{array} \right\} = \rho_e(\ell'_u(\phi), x'_e(\phi)).$$

Hence, the derivatives $(x'_e(\phi))_{e \in E'_\phi}$ together with the node labels $(\ell'_v(\phi))_{v \in V}$ form a thin flow with resetting on E^*_ϕ. □

As the second main result we show that the reverse direction of Theorem 3.6 is also true in the sense that we can use thin flows with resetting to construct a Nash flow over time. For this we first prove that there always exists a thin flow with resetting for any acyclic graph G' with arbitrary positive arc capacities and for any subset of resetting arcs E^*.

Theorem 3.7 (cf. Theorem 4 in [18]).
Consider an acyclic graph $G' = (V, E')$ with source s, sink t and capacities $(\nu_e)_{e \in E}$ as well as a subset of arcs $E^* \subseteq E'$. Suppose that every node $v \in V$ is reachable from s. Then there exists a thin flow $((x'_e)_{e \in E}, (\ell'_v)_{v \in V})$ with resetting on E^*.

The first proof in [18, Theorem 4] is quite complicated and uses some existence results of finite-dimensional variational equalities and the corresponding nonlinear complementarity problem. The proof we present in the following is due to [17, Theorem 3] and uses a set-valued function in order to apply Kakutani's fixed-point theorem.

Proof. Let X be the compact, convex and non-empty set of all static s-t-flows of value 1 and let $\Gamma\colon X \to 2^X$ be defined by

$$x' \mapsto \{\, y' \in X \mid y'_e = 0 \text{ for all } e = uv \in E' \text{ with } \ell'_v < \rho_e(\ell'_u, x'_e) \,\},$$

where $(\ell'_v)_{v \in V}$ are the node labels associated with x' uniquely defined by

$$\ell'_v = \begin{cases} \frac{1}{r} & \text{if } v = s, \\ \min\limits_{e = uv \in E'} \rho_e(\ell'_u, x'_e) & \text{if } v \in V \backslash \{\, s \,\}. \end{cases} \tag{3.8}$$

In order to use Kakutani's fixed point theorem (Theorem 2.9 on page 22) we show that all conditions are satisfied:

- The set $\Gamma(x')$ is non-empty, because there has to be at least one path P from s to t with $\ell'_v = \rho_e(\ell'_u, x'_e)$ for each arc e on P. If we set $y_e = 1$ for all arcs e on P and set every other value to 0 we obtain an element in $\Gamma(x')$.
- To see that $\Gamma(x')$ is convex, note that the arcs that can be used for sending flow, i.e., the ones satisfying $\ell'_v = \rho_e(\ell'_u, x'_e)$, are fixed within the set $\Gamma(x')$. Furthermore, every convex combination y of two elements $y^1, y^2 \in \Gamma(x')$ only uses arcs that are also used by y^1 or y^2.
- In order to show that the function graph $\{\, (x', y') \mid y' \in \Gamma(x) \,\}$ is closed let $(x^n, y^n)_{n \in \mathbb{N}}$ be a sequence within this set, i.e., $y^n \in \Gamma(x^n)$. Since both sequences, $(x^n)_{n \in \mathbb{N}}$ and $(y^n)_{n \in \mathbb{N}}$, are

contained in the compact set X they both have a limit x^* and y^* within X. Let $(\ell^n)_{n\in\mathbb{N}}$ be the sequence of associated node labels of (x^n) and ℓ^* the node label of x^*. Note that the mapping $x' \mapsto \ell'$ is continuous, and therefore, it holds that $\ell^* = \lim_{n\to\infty} \ell^n$. We prove that $y^* \in \Gamma(x^*)$. Suppose there is an arc $e = uv \in E'$ with $y^*_e > 0$ and $\ell^*_v < \rho_e(\ell^*_u, x^*_e)$. But since ρ_e is continuous there has to be an $n_0 \in \mathbb{N}$ such that $y^n_e > 0$ and $\ell^n_v < \rho_e(\ell^n_u, x^n_e)$ for all $n \geq n_0$, which is a contradiction. Hence, $\{ (x', y') \mid y' \in \Gamma(x) \}$ is closed.

Since all conditions for Kakutani's fixed point theorem are satisfied, there has to be a fixed point x^* of Γ. Let ℓ^* be the corresponding node labeling. We show that the pair (x^*, ℓ^*) satisfies the thin flow conditions. Equations (TF1) and (TF2) follow immediately by (3.8). For every arc $e = uv \in E'$ with $x^*_e > 0$ it holds that $\ell^*_v = \rho_e(\ell^*_u, x^*_e)$ since $x^* \in \Gamma(x^*)$, which shows Equation (TF3). Thus, (x^*, ℓ^*) forms a thin flow with resetting, which completes the proof. □

Finally, we show that thin flows are unique in the sense that the node labels are uniquely determined. Note that this is the best we can get as it is easy to see that the static flow x' is not unique in general. Consider a network consisting of two disjoint paths from s to t, where all arc capacities are greater than the network inflow rate. If the set of resetting arcs is empty then every static flow, no matter how we split it along the two paths, is a thin flow with resetting. Hence, x' is highly non-unique in this example. Nonetheless, it is easy to see that the node labels are uniquely determined in this specific network, as we have $\ell'_v = 1$ for all nodes v. In the following we show that this is no coincidence but holds in general.

Proposition 3.8 ([17, Theorem 4]). *The node labels of thin flows are uniquely determined by the network.*

Proof. Consider two thin flows (x', ℓ') and (y', k') on the same network. Note that the notation δ^- and δ^+ refers to all incoming and outgoing arcs in the graph $G' = (V, E')$. Let

$$W := \{ v \in V \mid \ell'_v > k'_v \}$$

and assume for contradiction that $W \neq \emptyset$.
For $e = uv \in \delta^+_W$ we have $x'_e \leq y'_e$, since otherwise we would have $x'_e > y'_e \geq 0$, and therefore, by (TF3) it would follow that

$$\ell'_v = \rho(\ell'_u, x'_e) > \rho(k'_u, y'_e) \geq k'_v.$$

This would be a contradiction to $v \notin W$. But we also obtain that $x'_e \geq y'_e$ for all $e = uv \in \delta^-_W$ since otherwise we would have $y'_e > x'_e \geq 0$ and again by (TF3) this would imply

$$k'_v = \rho(k'_u, y'_e) \geq \rho(\ell'_u, x'_e) \geq \ell'_v,$$

contradicting $v \in W$. Since both x' as well as y' are static flows of value 1 we have

$$\sum_{e\in\delta^+_W} x'_e - \sum_{e\in\delta^-_W} x'_e = \sum_{e\in\delta^+_W} y'_e - \sum_{e\in\delta^-_W} y'_e.$$

Hence, it follows that $x'_e = y'_e$ for all $e \in \delta^+_W \cup \delta^-_W$, since a strict inequality for some e could not be compensated. Furthermore, we have $x'_e = y'_e = 0$ for all $e \in \delta^-_W$ since otherwise we would have $y'_e > 0$ and as before this would lead to a contradiction to $v \in W$ as this would imply

$$k'_v = \rho(k'_u, y'_e) \geq \rho(\ell'_u, x'_e) \geq \ell'_v.$$

Since G' is acyclic and $s \notin W$, as $\ell'_s = k'_s = \frac{1}{r}$, there has to be a node $v^* \in W$ with $\delta^-_{v^*} \subseteq \delta^-_W$ and $\delta^-_{v^*} \neq \emptyset$ (this node can be obtained by starting at some node $v_0 \in W$ and then going backwards along an arc $v_1 v_0$ whenever $v_1 \in W$). For all arcs $e = uv^* \in \delta^-_{v^*}$ we have $x'_e = 0$, and therefore, $\ell'_{v^*} > k'_{v^*}$ implies $e \in E' \setminus E^*$. Hence, $\rho_e(\ell'_u, x'_e) = \ell'_u$ and $\rho_e(k'_u, y'_e) = k'_u$. But this is a contradiction to $v^* \in W$ since with (TF2) we have

$$k'_{v^*} = \min_{e=uv^* \in \delta^-_{v^*}} k'_u \geq \min_{e=uv^* \in \delta^-_{v^*}} \ell'_u = \ell'_{v^*}.$$

Thus, W is empty, implying $\ell'_v \leq k'_v$ for all $v \in V$ and for symmetric reasons it holds with the same argument that $\ell'_v \geq k'_v$ for all $v \in V$. This proves that the node labels are uniquely determined for all thin flows with resetting. □

3.5 Constructing Nash Flows Over Time

In this section we will show how to use a sequence of thin flows with resetting in order to construct a Nash flow over time.

Note that in a dynamic equilibrium no particle is able to overtake any other particle, and therefore, the choice of strategy for ϕ only depends on the strategies of flow that has entered the network before ϕ. So we may assume that the particles decide in order of their arrival at the source, i.e., ϕ_1 chooses a strategy before every $\phi_2 > \phi_1$. Due to this observation, it is possible to extend a given Nash flow over time up to some $\phi \in \mathbb{R}_{\geq 0}$ by using a thin flow on the current shortest paths network G'_ϕ with resetting on E^*_ϕ.

Restricted Nash flows over time. A **restricted Nash flow over time** on $[0, \phi]$ is a Nash flow over time where only the particles in $[0, \phi]$ are considered, i.e., for each arc $e = uv \in E$ we have $f^+_e(\theta) = 0$ for all $\theta > \ell_u(\phi)$, $f^+_e(\theta) = 0$ for all $\theta > \ell_v(\phi)$, and the net outflow at s has to be 0 for $\theta > \ell_s(\phi)$. In addition, the Nash flow condition (N) only must be satisfied for almost all times in $[0, \ell_u(\phi)]$.

α-Extension. Since all previous results carry over to restricted Nash flows over time, the earliest arrival times $(\ell_v)_{v \in V}$ are well-defined for particles in $[0, \phi)$, and therefore, it is possible to determine $G'_\phi = (V, E'_\phi)$ and E^*_ϕ; see Lemma 3.4. In order to extend a restricted Nash flow over time we first compute a thin flow on G'_ϕ with resetting on E^*_ϕ, and then extend the labels linearly as follows. For some $\alpha > 0$ we define

$$\ell_v(\phi + \xi) := \ell_v(\phi) + \xi \cdot \ell'_v \qquad \text{and} \qquad x_e(\phi + \xi) := x_e(\phi) + \xi \cdot x'_e.$$

for all $v \in V$, $e \in E$ and $\xi \in [0, \alpha]$.

Based on this we can extend the in- and outflow functions, i.e., for all $e = uv \in E$ we define

$$f^+_e(\theta) := \frac{x'_e}{\ell'_u} \quad \text{for } \theta \in [\ell_u(\phi), \ell_u(\phi + \alpha)) \qquad \text{and} \qquad f^-_e(\theta) := \frac{x'_e}{\ell'_v} \quad \text{for } \theta \in [\ell_v(\phi), \ell_v(\phi + \alpha)).$$

Note that in the case of $\ell'_u = 0$ the time interval is empty. This extended flow over time is called **α-extension** and we will show in Theorem 3.10 below that it is a restricted Nash flow over time on $[0, \phi + \alpha]$ as long as the α stays within the following bounds:

$$\ell_v(\phi) - \ell_u(\phi) + \alpha(\ell'_v - \ell'_u) \geq \tau_e \qquad \text{for all } e = uv \in E^*, \tag{3.9}$$

$$\ell_v(\phi) - \ell_u(\phi) + \alpha(\ell'_v - \ell'_u) \leq \tau_e \qquad \text{for all } e = uv \in E \setminus E'. \tag{3.10}$$

The first inequality ensures that no flow can traverse an arc faster than its transit time. It holds with equality when the queue of e depletes at time $\ell_u(\phi+\alpha)$, which we call **depletion event**. The second inequality makes sure that all non-active arcs are unattractive for all particles in $[\phi,\phi+\alpha)$. When it holds with equality the arc e becomes active for $\phi+\alpha$ and we speak of an **active event**. Whenever one of these events occurs we must compute a new thin flow with resetting because either a resetting arc has become non-resetting or a non-active arc has become active. It is easy to see that there exists an $\alpha>0$ that satisfies these inequalities since $\ell_v(\phi)>\ell_u(\phi)+\tau_e$ for arcs $e\in E^*_\phi$ and $\ell_v(\phi)<\ell_u(\phi)+\tau_e$ for arcs $e\notin E'_\phi$ as stated in Lemma 3.4.
For the maximal α we call the interval $[\phi,\phi+\alpha)$ **thin flow phase**.

Lemma 3.9. *The α-extension forms a feasible flow over time and the extended ℓ-labels coincide with the earliest arrival times, i.e., they satisfy* (3.4) *for all $\varphi\in[\phi,\phi+\alpha)$.*

The flow conservation follows immediately from the flow conservation of x' and 3.4 can be proven by distinguishing three cases. If the arc is non-active it stays non-active during the extended interval. For active, but non-resetting arcs that do not build up a queue, we obtain $\ell_v(\phi+\xi)\le T_e(\ell_u(\phi+\xi))$ from (TF2), with equality if $\ell'_v=\rho_e(\ell'_u,x'_e)$. The same is true for resetting arcs or arcs that build up a queue, even though the proof for this is a bit more technical. A detailed proof is given in the appendix on page 50.

Theorem 3.10 (cf. Theorem 3 [55]).
Given a restricted Nash flow over time $f=(f^+_e,f^-_e)_{e\in E}$ on $[0,\phi)$ and $\alpha>0$ satisfying (3.9) *and* (3.10)*, the α-extension is a restricted Nash flow over time on $[0,\phi+\alpha)$.*

Proof. Lemma 3.3 yields $F^+_e(\ell_u(\varphi))=F^-_e(\ell_v(\varphi))$ for all $\varphi\in[0,\phi)$, so for $\xi\in[0,\alpha)$ it holds that

$$F^+_e(\ell_u(\phi+\xi))=F^+_e(\ell_u(\phi))+\frac{x'_e}{\ell'_u}\cdot\xi\cdot\ell'_u=F^-_e(\ell_v(\phi))+\frac{x'_e}{\ell'_v}\cdot\xi\cdot\ell'_v=F^-_e(\ell_v(\phi+\xi)).$$

By Lemmas 3.3 and 3.9 the α-extension is a restricted Nash flow over time on $[0,\phi+\alpha)$. □

Finally, we show that this construction leads to an unrestricted Nash flow over time.

Theorem 3.11.
There exists a Nash flow over time.

Proof. The process starts with the empty flow over time, i.e., a restricted Nash flow over time for $[0,0)$. We apply Theorem 3.10 iteratively and choose α maximal according to (3.9) and (3.10). If one of the α is unbounded we are done. Otherwise, we obtain a sequence $(f_i)_{i\in\mathbb{N}}$, where f_i is a restricted Nash flow over time for $[0,\phi_i)$, with a strictly increasing sequence $(\phi_i)_{i\in\mathbb{N}}$. In the case that this sequence has a finite limit, say $\phi_\infty<\infty$, we define a restricted Nash flow over time f^∞ for $[0,\phi_\infty)$ by using the point-wise limit of the x- and ℓ-labels, which exists due to monotonicity and boundedness of these functions. Note that there are only finitely many different thin flows, and therefore, the derivatives x' and ℓ' are bounded. Then the process can be restarted from this limit point. Let $\mathcal{P}_G$ be the set of all particles $\phi\in\mathbb{R}_{\ge0}$ for which there exists a restricted Nash flow over time on $[0,\phi)$ constructed as described above. The set $\mathcal{P}_G$ cannot have a maximal element because the corresponding Nash flow over time could be extended by using Theorem 3.10. But $\mathcal{P}_G$ cannot have an upper bound either since the limit of any convergent sequence would be contained in this set. Therefore, there exists an unbounded increasing sequence $(\phi_i)_{i=1}^\infty\in\mathcal{P}_G$. As thin flows are unique we have that the ℓ-functions coincide on $[0,\phi_i]$ for all restricted thin flows up to ϕ_j with $j\ge i$. The same property can be obtained for the x-function by setting $x_j(\phi) := x_i(\phi)$ for all $j \geq i$. Hence, we can construct a Nash flow over time f on $[0, \infty)$ by taking the point-wise limit of the x- and ℓ-labels, completing the proof. □

Example. An example of a Nash flow over time with four thin flow phases is shown in Figure 3.3 on the next page. In this specific network the center arc uv builds up a queue during the time interval $[4, 5]$, which then decreases afterwards and depletes at time 6. This causes a depletion event for particle $\phi = 3$. Interestingly, this arc even leaves the current shortest paths network for particles strictly greater than 6 as ℓ_u increases faster than ℓ_v.

3.6 Further Results

In the following we present some additional results for the base model.

3.6.1 Mixed Integer Programs for Thin Flow Computations

Even though Theorem 3.7 guarantees the existence of a thin flow, it does not give clear instructions on how to compute it. In fact, it is an open question whether this computation can be done efficiently or not. We want to present a mixed integer program that outputs a thin flow with resetting. Even though this is not efficient theoretically, experiments showed, that a Nash flow over time in a network of medium size can be computed rather quickly with a fast MIP-solver.
Besides the continuous variables x' and ℓ' we introduce two types of boolean variables. For each $e = uv \in E' \setminus E^*$ we define $a_e \in \{0, 1\}$ to be 1 if $\ell'_v > \rho_e(x'_e, \ell'_u)$. Hence, in this case we require that $x'_e = 0$. Note that we do not need this boolean variable for an arc $e = uv \in E^*$ since we have $\ell'_v = \rho_e(x'_e, \ell'_u) = \frac{x'_e}{\nu_e}$ regardless whether $x'_e = 0$ or $x'_e > 0$. The variable $b_e \in \{0, 1\}$ indicates with $b_e = 1$ for an arc $e = uv \in E' \setminus E^*$ that $\ell'_u \geq \frac{x'_e}{\nu_e}$. Let $B > 0$ be a number that is greater than the maximal possible value for any ℓ'_v or x'_e plus 1. Then we obtain the following constraints:

$$
\begin{aligned}
\ell'_s &= \tfrac{1}{r} && \\
\ell'_v &= \tfrac{x'_e}{\nu_e} && \text{for all } e = uv \in E^*, \\
&\left.\begin{aligned}
\ell'_v &\leq \tfrac{x'_e}{\nu_e} + b_e \cdot B \\
\ell'_v &\leq \ell'_u + (1 - b_e) \cdot B \\
\ell'_v &\geq \tfrac{x'_e}{\nu_e} - a_e \cdot B \\
\ell'_v &\geq \ell'_u - a_e \cdot B \\
x'_e &\leq (1 - a_e) \cdot B \\
\tfrac{x'_e}{\nu_e} - \ell'_u &\leq 1 - b_e \\
\ell'_u - \tfrac{x'_e}{\nu_e} &\leq b_e
\end{aligned}\right\} && \text{for all } e = uv \in E' \setminus E^*.
\end{aligned}
$$

The third and forth equation guarantee that $\ell'_v \leq \rho_e(x'_e, \ell'_u) = \max\{\ell'_u, \frac{x'_e}{\nu_e}\}$ for all $e \in E' \setminus E^*$ and the fifth and sixth equation ensure that this holds with equality if $a_e = 0$. The last three equations implement that $x'_e = 0$ if $a_e = 1$ as well as $\ell'_u \geq \frac{x'_e}{\nu_e}$ for $b_e = 1$ and $\ell'_u \leq \frac{x'_e}{\nu_e}$ otherwise. Together with the region constraints

$$0 \leq \ell'_v \leq B - 1, \qquad 0 \leq x'_e \leq B - 1, \qquad a_e \in \{0, 1\} \quad \text{and} \quad b_e \in \{0, 1\}$$

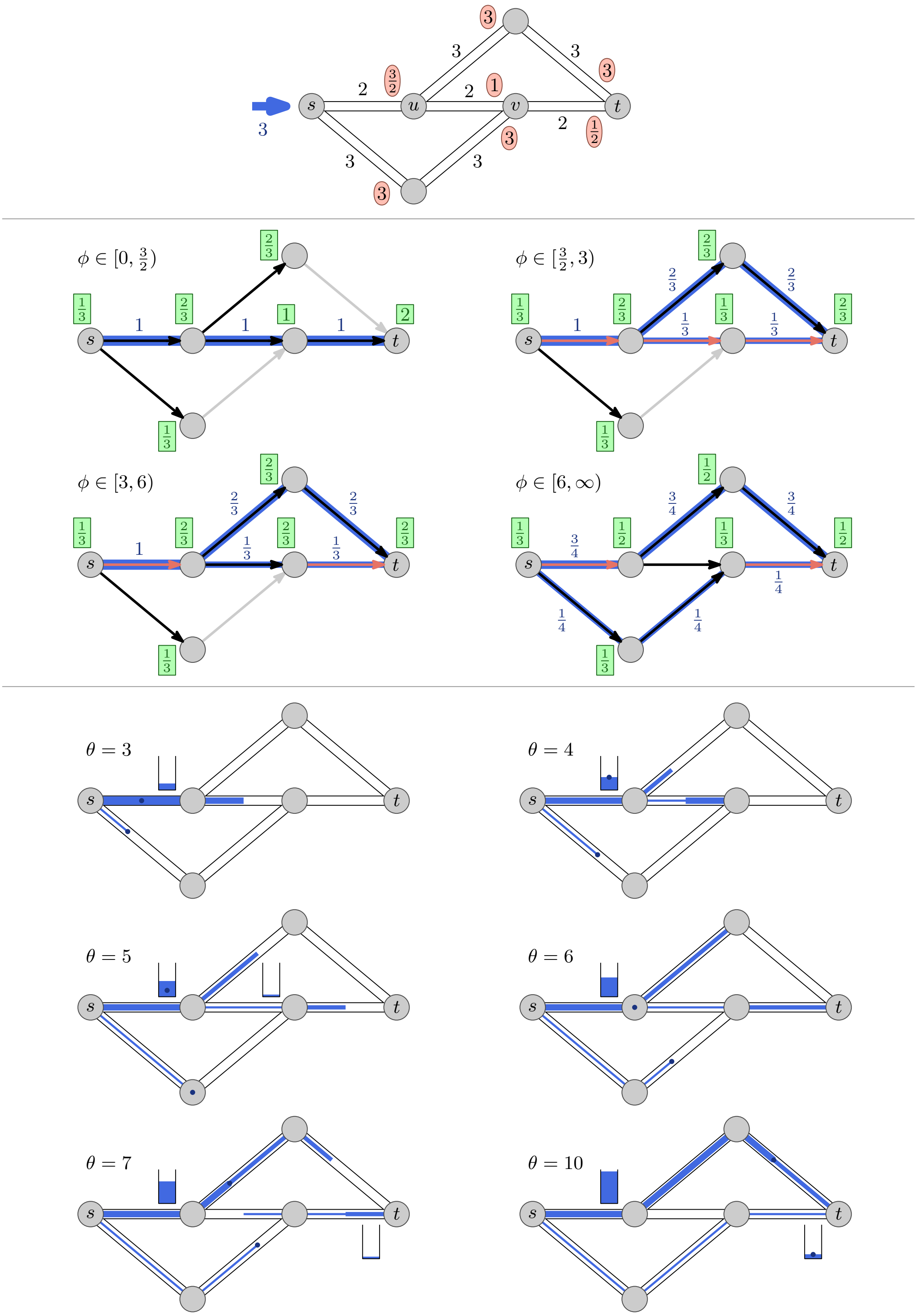


Figure 3.3: An example of a Nash flow over time with corresponding thin flows. The network inflow rate as well as transit times and capacities (circled) are displayed *on top*. We have four thin flow phases, which are depicted *in the middle*. Here, black arcs are active but non-resetting (at the beginning of each phase), red arcs are resetting and gray arcs are inactive. The arc labels x' and the node labels ℓ' (framed) are shown close to the arc/node. *On the bottom part* the resulting Nash flow over time is displayed at various snapshots in time. Here, the small disc shows the position of particle $\phi = 6$ along the top and bottom path.

this ensures that a feasible point is a thin flow with resetting. Note that this mixed integer program does not have any objective function and the difficulty is not to find an optimum but to find a feasible point instead. But unfortunately, finding a feasible solution of a mixed integer program is in general an NP-hard problem as well.

3.6.2 Uniqueness of Earliest Arrival Times

It is worth mentioning that, since the node labels of thin flows are unique, the earliest arrival time functions coincide for all Nash flows over time that are constructed as described above. More precisely, all earliest arrival time functions $(\ell_v)_{v \in V}$ of Nash flows over time f with right-continuous in- and outflow rates are the same for a given network; see [17, Theorem 6]. It is an open question though whether this is true for *all* Nash flows over time on a given network. There might be the case that for some particle ϕ the thin flow $(x'(\phi), \ell'(\phi))$ gives a clear instruction for all particles in $[\phi, \phi + \varepsilon]$. But instead that all those particles proceeds as this thin flow suggests, there might additionally be a decreasing sequence of event points ϕ_i together with thin flows $(x'(\phi_i), \ell'(\phi_i))$ for all $\phi \in [\phi_i, \phi_{i-1})$ such that $\phi_i \searrow \phi$. In the case that successive thin flows in this sequence differ, we would obtain a Nash flow over time with different earliest arrival times. Such a behavior would imply some very counter-intuitive effects, for example, some thin flows would occur an infinite number of times within one Nash flow over time meaning that on some arcs a queue would grow and deplete indefinitely. It would also imply that Nash flows over time are very unstable in general and that small perturbations could have major effects. As we have not seen any example of such a *butterfly effect* in Nash flows over time we conjecture that this does not happen and that the earliest arrival times are unique in general. But since we do not have a proof for this, it remains an open question.

3.6.3 Prices of Anarchy

The question of a bound on the price of anarchy for Nash flows over time was first raised by Koch and Skutella in [54]. They showed that in shortest paths networks the price of anarchy is 1. This means that in networks where every arc is active for $\phi = 0$ implying that the first thin flow phase never ends, Nash flows over times are optimal and equal earliest arrival flows. In 2015, Bhaskar, Fleischer and Anshelevich [5] characterized essentially two kinds of prices of anarchy, which we want to briefly present here.

Suppose we have a fixed amount of flow volume $A \geq 0$ at the source. If we take the ratio of the arrival time of the last particle $\phi_0 = A$ in the worst Nash flow over time to the completion time of a quickest flow (or earliest arrival flow), we obtain the **time price of anarchy** (also called **makespan-PoA**).

Similarly, we can consider a fixed time horizon H and then compare the total volume of a maximum flow to the total volume of particles in $\{ \phi \in \mathbb{R}_{\geq 0} \mid \ell_t(\phi) \leq H \}$ (which equals $\ell_t^{-1}(H)$) of a worst Nash flow over time. The ratio of these two is called the **evacuation price of anarchy** (also **throughput-PoA**).

There are even more prices of anarchy that can be considered in this model, for example, if we consider not only the arrival of the last particle but the *average* arrival time of all particles. If we compare this value of an earliest arrival flow to the average arrival time in the worst Nash flow over time for a fixed amount of flow volume, then we obtain the **total delay price of anarchy**. Analogously, we can consider the *average* flow amount that has arrived at t over time. The ratio of this value of the worst Nash flow over time to an earliest arrival time up to some fixed time horizon H is called **work price of anarchy**. As they share most properties with the time and evacuation price of anarchy, up to some constant factor, we will not include the total delay or work price of anarchy into further consideration.

It was shown that the evacuation price of anarchy lies in $\Omega(|E|)$, and therefore, is unbounded for arbitrary networks [54]. For this reason we focus on the time price of anarchy for the remaining part of this section.

Bhaskar et al. showed in [5] that it is possible to obtain a bound on the time price of anarchy of $\frac{e}{e-1}$ if we are allowed to reduce the capacities of all arcs in the network before constructing a Nash flow over time. Note that the completion time of the quickest flow is still determined in the original network. The key idea here is to compute a quickest flow, which is a temporary repeated flow (see *maximum flows over time* on page 24), and then to reduce the capacities to the flow values of the underlying static flow. This ensures that in the evolution of the Nash flow over time no arc will ever leave the current shortest paths network. Instead, the Nash flow over time uses the exact same paths as the quickest flow, but some of the paths will be delayed due to congestion. But this delay can be bounded in these specific networks such that the time price of anarchy is bounded as desired.

Very recently, Correa, Cristi and Oosterwijk [20] extended this result and showed that the time price of anarchy is bounded by $\frac{e}{e-1}$ whenever the network inflow rate r does not exceed the initial network inflow rate r' of a quickest flow (that does not build up any queues). Here, r' equals the flow value of the static flow underlying the corresponding temporary repeated flow. This is a massive improvement compared to the result of Bhaskar et al. as now it is only necessary to reduce the network inflow rate (so to say, the capacity of only *one* arc) instead of modifying the whole network. The key idea is to consider the difference between the arrival time of the last particle and the completion time of a quickest flow. It turns out that under the network inflow rate requirement we have

$$\ell_t(\hat{\phi}) - \mathrm{OPT} \le \frac{1}{r} \cdot \sum_{e=uv\in E} z_e(\ell_u(\hat{\phi})),$$

where $\hat{\phi}$ is the last particle that is sent into the network.

By some thoughtful calculations it is possible to bound the total amount of queues seen by some particle ϕ by the total increase of the sink arrival time during $[0,\phi]$. More precisely, it holds for all $\phi \in \mathbb{R}_{\ge 0}$ that

$$\sum_{e=uv\in E} z_e(\ell_u(\phi)) \le \frac{r}{e} \cdot (\ell_t(\phi) - \ell_t(0)).$$

Using this for the last particle $\hat{\phi}$ we obtain for the time price of anarchy that

$$\frac{\ell_t(\hat{\phi})}{\mathrm{OPT}} \le \frac{e}{e-1}.$$

Correa et al. also showed in the same publication that this bound is in fact tight for the family of all considered networks with the condition on the network inflow rate.

It is natural to conjecture that $\frac{e}{e-1}$ is also the tight bound for the general case as the following quite natural monotonicity condition would imply this.

Conjecture 3.12. *Consider the same network but two different network inflow rates $r_1 < r_2$ and, for both, a Nash flow over time f^1 and f^2. Then we have $\ell^1_t(\hat{\phi}) \ge \ell^2_t(\hat{\phi})$.*

Intuitively, this means that if we have the same network but we increase the network inflow rate then we expect that the last particle should not arrive later than before. This seems very plausible since the last particle enters the network strictly earlier, and intuitively, all particles in front had more time to travel towards the sink. But as the flow dynamics of a Nash flow over time are quite unpredictable as soon as the thin flow phases change, this conjecture remains an open problem and with it a bound on the time price of anarchy for general networks.

3.6.4 Long-Term Behavior

As we have seen, a sequence of thin flows is a perfect description of a Nash flow over time when we consider it chronologically. But from an application perspective the initial traffic evolution might not be that important and it might be more interesting to understand the long-term behavior and the question whether the traffic dynamics will settle down to a steady state. And it would be even better if we could skip the initial phases altogether as they are very hard to compute, and instead, directly characterize how such a steady state could look like. Cominetti, Correa and Olver found out that this is indeed possible and that the steady state, if it exists, can be computed efficiently by a linear program and its dual [19]. They also proved that the question, whether such a steady state exists, can easily be determined by considering the minimal s-t-cut (with respect to the capacities) in the network.

To be more precise, we define the **steady state** as the thin flow phase with thin flow (x', ℓ'), where $\ell'_v = \frac{1}{r}$ for all $v \in V$. Hence, if the Nash flow over time ever reaches such a phase then the arrival times will all grow with a constant slope, which means that all travel times stay constant. But this immediately implies that this phase lasts indefinitely as there cannot occur new events anymore. In this phase all queue lengths (and therefore all waiting times) stay constant.

Cominetti et al. show that such a steady state exists and that it is reached in finite time if, and only if, the network inflow rate does not exceed the minimal cut of the network. It is easy to see that this is a necessary condition because, if the network inflow rate is strictly larger than the minimal cut, the sum of the queues on all arcs of the cut must increase unlimited, and hence, there cannot be a steady state.

The other direction is a bit more involved. The key idea is that the following potential function

$$\Phi(\phi) := r \cdot (\ell_t(\phi) - \ell_s(\phi)) - \sum_{e \in E} z_e(\ell_v(\phi))$$

is strictly increasing with strictly positive derivatives unless the Nash flow over time has reached a steady state. In addition, this potential function is bounded from above, and hence, the Nash flow over time reaches a steady state in finite time.

It is worth noting though, that this does not mean that there are only finitely many thin flow phases in those networks. In principle, it could be possible that there are infinitely many phases with decreasing length and that the event points converge to some $\phi \in \mathbb{R}_{\geq 0}$. Even though we have never seen such an example and we conjecture that this does not happen, the question whether there are only finitely many phases remains open.

In their publication, Cominetti et al. also proved that a steady state is a solution to the following pair of primal-dual linear programs.

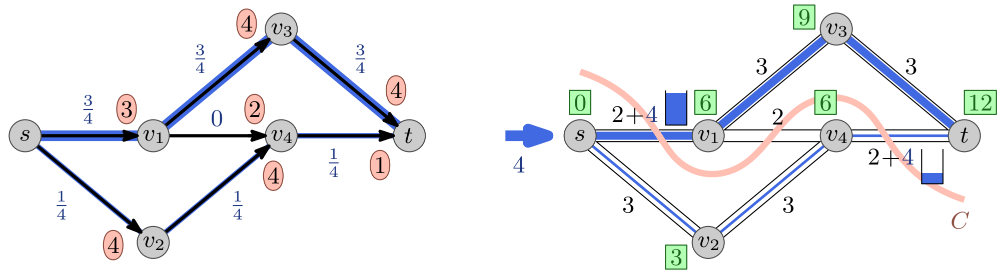


Figure 3.4: A steady state represented by a prim-dual-solution. *On the left:* The thin flow x' together with the capacities ν_e (cycled) for each arc e. Note that the node labels ℓ' are omitted as they are 1 at every node in every steady state. *On the right:* The travel times d_v (framed) for every node v as well as the transit times τ_e and waiting times q_e (if positive) for each arc e.
Note that for each arc $e = uv$ with $x'_e > 0$ the condition $d_v \leq d_u + \tau_e + q_e$ is tight. For the central arc $e = v_1v_4$ we have $x'_e = 0$ and $d_{v_4} < d_{v_1} + \tau_e + q_e$. The steady state is not unique as all waiting times along the s-t-cut $C = \{sv_1, v_4t\}$ could be uniformly reduced until the condition for the central arc v_1v_4 is tight ($q_{sv_1} = q_{v_4t} = 2$) or, alternatively, they could be increased by an arbitrary amount.

Primal:

$$\begin{aligned} \min_{x' \in \mathbb{R}^{|E|}} \quad & \sum_{e \in E} \tau_e \cdot x'_e \\ \text{s.t.} \quad & x' \in \mathcal{F}, \\ & 0 \leq x'_e \leq \frac{\nu_e}{r} \qquad \text{for all } e \in E, \end{aligned}$$

Dual:

$$\begin{aligned} \max_{\substack{d \in \mathbb{R}^{|V|} \\ q \in \mathbb{R}^{|E|}}} \quad & d_t - \sum_{e \in E} \frac{\nu_e}{r} \cdot q_e \\ \text{s.t.} \quad & d_s = 0, \\ & d_v \leq d_u + \tau_e + q_e \qquad \text{for all } e = uv \in E, \\ & q_e \geq 0 \qquad \text{for all } e \in E. \end{aligned}$$

Here, $\mathcal{F}$ denotes the set of all static s-t flows of value 1 described by

$$\sum_{e \in \delta_v^+} x'_e(\phi) - \sum_{e \in \delta_v^-} x'_e(\phi) = \begin{cases} 0 & \text{if } v \in V \setminus \{s, t\}, \\ 1 & \text{if } v = s, \\ -1 & \text{if } v = t, \end{cases}$$

and d_v corresponds to the travel time from s to v, i.e., $d_v = \ell_v(\phi) - \ell_s(\phi)$. These travel times d_v as well as the queues q_e are constant in a steady state.
We want to give an intuition, why an optimal solution of this prime-dual linear program corresponds to a thin flow x' (primal) and travel times and queues (dual) of a steady state in a Nash flow over time; see Figure 3.4.
First of all, in a steady state the thin flow x'_e must be a static s-t-flow of value 1 that satisfies $x'_e \leq \frac{\nu_e}{r}$, since otherwise $\ell'_v = \rho_e(\ell'_u, x'_e) \geq \frac{x'_e}{\nu_e} > \frac{1}{r}$, which is not possible in the steady state where $\ell'_v = \frac{1}{r}$. Furthermore, queues must always be non-negative and the travel time d_s from s to s must be 0. For all $e = uv \in E$ Equation (3.4) implies that

$$d_v = \ell_v(\phi) - \ell_s(\phi) \leq \ell_u(\phi) + \tau_e + q_e(\phi) - \ell_s(\phi) = d_u + \tau_e + q_e.$$

Additionally, we can observe that the complementary slackness corresponds to the thin flow and Nash flow conditions: If $x'_e > 0$ we know that (TF3) holds, which is equivalent to the condition that e must be active during the steady state phase, i.e., the equation above holds with equality. For $x'_e < \frac{\nu_e}{r}$ it must be the case that there is no queue on this arc, i.e., $q_e = 0$, since otherwise this queue would decrease, which is not allowed for a steady state.

Note that this linear program formulation enables us to compute steady states efficiently, but it turns out that the queues are not uniquely defined in general. In the case that there is an s-t-cut C only consisting of resetting arcs (or more precisely arcs with $x'_e = \frac{\nu_e}{r}$), we obtain a new solution, and therefore a new steady state, by adding any $a > 0$ to q_e for each arc $e \in C$. Note that the objective value of the dual stays constant as d_t increases by a and the sum increases by

$$\sum_{e \in C} a \cdot \frac{\nu_e}{r} = a \cdot \sum_{e \in C} x'_e = a.$$

Hence, this linear program does not help us to determine which of these steady states the Nash flow over time reaches in the long-term. For more details on this topic we refer to the publication of Cominetti, Correa and Olver [19].

3.7 Appendix: Technical Proofs

Lemma 3.1. *For a feasible flow over time f it holds for all $e \in E$, $v \in V$ and $\theta \in [0, \infty)$ that:*

(i) $q_e(\theta) > 0 \quad \Leftrightarrow \quad z_e(\theta + \tau_e) > 0.$

(ii) $z_e(\theta + \tau_e + \xi) > 0$ *for all* $\xi \in [0, q_e(\theta))$.

(iii) $F_e^+(\theta) = F_e^-(T_e(\theta))$.

(iv) For $\theta_1 < \theta_2$ *with* $F_e^+(\theta_2) - F_e^+(\theta_1) = 0$ *and* $z_e(\theta_2 + \tau_e) > 0$ *we have* $T_e(\theta_1) = T_e(\theta_2)$.

(v) The functions T_e *are monotonically increasing.*

(vi) The functions F_e^+, F_e^-, z_e, q_e *and* T_e *are almost everywhere differentiable.*

(vii) For almost all $\theta \in [0, \infty)$ *we have*

$$q_e'(\theta) = \begin{cases} \frac{f_e^+(\theta)}{\nu_e} - 1 & \text{if } q_e(\theta) > 0, \\ \max\left\{ \frac{f_e^+(\theta)}{\nu_e} - 1, 0 \right\} & \text{else.} \end{cases}$$

Proof.

(i) This follows directly by the definition of q_e.

(ii) By (3.3) we have that $f_e^-(\xi) \leq \nu_e$ almost everywhere. Since F_e^+ is monotonically increasing, we obtain for $0 \leq \xi < q_e(\theta)$ that

$$z_e(\theta+\tau_e+\xi) = F_e^+(\theta+\xi) - F_e^-(\theta+\tau_e+\xi) \geq F_e^+(\theta) - F_e^-(\theta+\tau_e) - \xi \cdot \nu_e > z_e(\theta+\tau_e) - q_e(\theta) \cdot \nu_e = 0.$$

(iii) Again by (3.3) together with (ii) we obtain for almost all $\xi \in [\theta + \tau_e, \theta + \tau_e + q_e(\theta))$ that $f_e^-(\xi) = \nu_e$. Hence,

$$F_e^-(T_e(\theta)) = F_e^-(\theta + \tau_e) + q_e(\theta) \cdot \nu_e = F_e^-(\theta + \tau_e) + z_e(\theta + \tau_e) = F_e^+(\theta).$$

(iv) Intuitively, this holds true since whether a particle enters the queue at time θ_1 or θ_2 does not influence the exit time, as long as no other flow enters the queue during the interval $[\theta_1, \theta_2]$ and as long as the queue does not deplete during this time. Formally, this follows since

$$z_e(\xi + \tau_e) = F_e^+(\xi) - F_e^-(\xi + \tau_e) \geq F_e^+(\theta_2) - F_e^-(\theta_2 + \tau_e) = z_e(\theta_2 + \tau_e) > 0,$$

and therefore $f_e^-(\xi + \tau_e) = \nu_e$ for almost all $\xi \in [\theta_1, \theta_2]$. Thus,

$$\begin{aligned} T_e(\theta_1) &= \theta_1 + \tau_e + \frac{F_e^+(\theta_1) - F_e^-(\theta_1 + \tau_e)}{\nu_e} \\ &= \theta_1 + \tau_e + \frac{F_e^+(\theta_2) - F_e^-(\theta_2 + \tau_e) + (\theta_2 - \theta_1) \cdot \nu_e}{\nu_e} = T_e(\theta_2). \end{aligned}$$

(v) Consider two points in time $\theta_1 < \theta_2$. By (3.3) we have that $f_e^-(\xi) \leq \nu_e$ almost everywhere, and therefore $F_e^-(\theta_2) - F_e^-(\theta_1) \leq (\theta_2 - \theta_1) \cdot \nu_e$. Since F_e^+ is monotonically increasing we obtain

$$\begin{aligned} T_e(\theta_1) &= \theta_1 + \tau_e + \frac{F_e^+(\theta_1 - \tau_e) - F_e^-(\theta_1)}{\nu_e} \\ &\leq \theta_1 + \tau_e + \frac{F_e^+(\theta_2 - \tau_e) - F_e^-(\theta_2) + (\theta_2 - \theta_1) \cdot \nu_e}{\nu_e} = T_e(\theta_2). \end{aligned}$$

(vi) Since f_e^+ and f_e^- are locally integrable, Lebesgue's differentiation theorem (Theorem 2.4) yields that the integral functions F_e^+ and F_e^- are almost everywhere differentiable. As summation and scaling preserve this property we have that z_e, q_e and T_e are almost everywhere differentiable as well.

(vii) For almost all $\theta \in [0, \infty)$ we have by (3.3) that

$$z_e'(\theta + \tau_e) = f_e^+(\theta) - f_e^-(\theta + \tau_e) = \begin{cases} f_e^+(\theta) - \nu_e & \text{if } z_e(\theta + \tau_e) > 0, \\ \min\{\, 0, f_e^+(\theta) - \nu_e \,\} & \text{else.} \end{cases}$$

The claim follows immediately by using (i). □

Lemma 3.3 (cf. Theorem 1 in [17]). *Let f be a feasible flow over time, $\Phi_e := \{\, \phi \in \mathbb{R}_{\geq 0} \mid e \in E'_\phi \,\}$ the set of all particles for which e is active and $\Phi_e^c := \mathbb{R}_{\geq 0} \setminus \Phi_e$ its complement. Then the following statements are equivalent:*

(i) f is a Nash flow over time.

(ii) For each arc $e = uv$, it holds that $f_e^+(\theta) = 0$ for almost all $\theta \in \ell_u(\Phi_e^c)$.

(iii) $F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi))$ holds for all arcs $e = uv$ and all particles ϕ.

(iv) For every arc $e = uv$ and almost all $\phi \in \Phi_e^c$ we have $f_e^+(\ell_u(\phi)) \cdot \ell_u'(\phi) = 0$.

(v) For all ϕ and every arc $e = uv$ we have: If $F_e^+(\ell_u(\phi) - \varepsilon) < F_e^+(\ell_u(\phi))$ for all $\varepsilon > 0$ then $e \in E'_\phi$.

Proof. $(i) \Leftrightarrow (ii)$: Note that the flow conservation implies that $f_e^+(\theta) = 0$ for almost all $\theta \in [0, \ell_u(0))$ for an arc $e = uv$, since flow only starts at the source at time 0 and it cannot reach u faster than $\ell_u(0)$. Furthermore, we have $\ell_u(\theta) \geq \theta$, and therefore, ℓ_u is unbounded. In other words, ℓ_u is surjective on $[\ell_u(0), \infty)$.
The contraposition of the Nash flow condition reads $f_e^+(\theta) = 0$ for almost all $\theta \in \ell_u(\Phi_e)^c$. Hence, it is sufficient to show that for almost all $\xi \in [\ell_u(0), \infty)$ we have

$$\xi \in \ell_u(\Phi_e)^c \quad \Leftrightarrow \quad \xi \in \ell_u(\Phi_e^c).$$

"$\Rightarrow$": Let $\phi \in \mathbb{R}_{\geq 0}$ with $\ell_u(\phi) = \xi \in \ell_u(\Phi_e)^c$. It follows that $\phi \notin \Phi_e$, and hence, $\ell_u(\phi) \in \ell_u(\Phi_e^c)$.
"$\Leftarrow$": Let $\xi \in \ell_u(\Phi_e^c)$ and suppose $\xi \notin \ell_u(\Phi_e)^c$, i.e., there are two different particles $\phi_1 \in \Phi_e^c$ and $\phi_2 \in \Phi_e$ with $\ell_u(\phi_1) = \xi = \ell_u(\phi_2)$. Since ℓ_u is monotonically increasing, ℓ_u has to be constant between ϕ_1 and ϕ_2, and therefore, there exists a rational number $\kappa_\xi \in \mathbb{Q}$ with $\ell_u(\kappa_\xi) = \xi$. Since for every point in time $\xi \in \ell_u(\Phi_e^c)$ with $\ell_u(\Phi_e)^c$ there is a $\kappa_\xi \in \mathbb{Q}$ with $\ell_u(\kappa_\xi) = \xi$, the set $\ell_u(\Phi_e^c) \setminus \ell_u(\Phi_e)^c$ is a subset of $\ell_u(\mathbb{Q})$, and hence, it is countable, or in other words, a null set.

$(ii) \Leftrightarrow (iii)$: Fix an arc $e = uv$. For all $\phi \in \mathbb{R}_{\geq 0}$ let $I_\phi := (\phi_0, \phi]$, where $\phi_0 \in [0, \phi]$ is the maximal value with $T_e(\ell_u(\phi_0)) = \ell_v(\phi)$ or 0 if no such ϕ_0 exists. Note that $T_e(\ell_u(\phi_0)) > \ell_v(\phi)$ in the latter

case, since $T_e(\ell_u(\phi)) \geq \ell_v(\phi)$ holds in general and $T_e \circ \ell_u$ is continuous. In the case that $\phi \in \Phi_e$ we have $I_\phi = \emptyset$.
We show for $\phi' > 0$ that

$$\phi' \in \bigcup_{\phi>0} I_\phi \quad \Leftrightarrow \quad \phi' \in \Phi_e^c. \tag{3.11}$$

On the one hand, we have for all $\phi' \in \Phi_e^c \backslash \{0\}$ that $T_e(\ell_u(\phi')) > \ell_v(\phi')$, and therefore, there is a $\phi_0 < \phi'$ which implies $\phi' \in I_{\phi'}$. On the other hand, for all $\phi' \in I_\phi$ we have $\ell_v(\phi') < T_e(\ell_u(\phi'))$, which implies $\phi' \in \Phi_e^c$, because otherwise we would have, by the monotonicity of ℓ_v and $T_e \circ \ell_u$, that

$$T_e(\ell_u(\phi')) \leq \ell_v(\phi') \leq \ell_v(\phi) \leq T_e(\ell_u(\phi_0)) \leq T_e(\ell_u(\phi')),$$

and therefore equality, contradicting the maximality of ϕ_0. This finishes the proof of (3.11).
Hence, e is not active for all $\phi' \in (\phi_0, \phi]$. Furthermore, we have $F_e^+(\ell_u(\phi_0)) = F_e^-(T_e(\ell_u(\phi_0)) = F_e^-(\ell_v(\phi))$. Note that for $\phi_0 = 0$ we have $0 \geq F_e^-(\ell_v(\phi)) \geq F_e^+(T_e(\ell_u(\phi_0)) = 0$.
Suppose (ii) is given, which means $f_e^+(\theta) = 0$ for almost all $\theta \in \ell_u(I_\phi) = (\ell_u(\phi_0), \ell_u(\phi)]$. This yields

$$F_e^+(\ell_u(\phi)) - F_e^-(\ell_v(\phi)) = F_e^+(\ell_u(\phi)) - F_e^+(\ell_u(\phi_0)) = \int_{\ell_u(\phi_0)}^{\ell_u(\phi)} f_e^+(\theta)\,\mathrm{d}\theta = 0,$$

which shows (iii).
Conversely, suppose that (iii) holds. We have that Φ_e^c is a union of countably many intervals I_ϕ for which we have

$$\int_{I_\phi} f_e^+(\theta)\,\mathrm{d}\theta = F_e^+(\ell_u(\phi)) - F_e^+(\ell_u(\phi_0)) = F_e^+(\ell_u(\phi)) - F_e^-(\ell_v(\phi)) = 0,$$

which proves (ii).

$(ii) \Leftrightarrow (iv)$: For every arc $e = uv$ the rule of integration by substitution with $\theta = \ell_u(\phi)$ yields

$$\int_{\ell_u(\Phi_e^c)} f_e^+(\theta)\,\mathrm{d}\theta = \int_{\Phi_e^c} f_e^+(\ell_u(\phi)) \cdot \ell_u'(\phi)\,\mathrm{d}\phi.$$

So this either equals 0 or not, which shows that (ii) is equivalent to $f_e^+(\ell_u(\phi)) \cdot \ell_u'(\phi) = 0$ for almost all $\phi \in \Phi_e^c$, i.e., equivalent to (iv).

$(i) \Rightarrow (v)$: Suppose we have $F_e^+(\ell_u(\phi) - \varepsilon) < F_e^+(\ell_u(\phi))$ for all $\varepsilon > 0$. Since $F_e^+(\ell_u(\phi)) > 0$ the Nash flow condition implies that e was part of the current shortest paths network at some point in time before ϕ. Let $\phi' \leq \phi$ be the last point in time with $e \in E'_{\phi'}$. Since e was not in the current shortest paths network in-between ϕ' and ϕ there is no inflow during $[\ell_u(\phi'), \ell_u(\phi)]$, i.e., $F_e^+(\ell_u(\phi)) - F_e^+(\ell_u(\phi')) = 0$. Since we assume that $F_e^+(\ell_u(\phi) - \varepsilon) < F_e^+(\ell_u(\phi))$ this implies that $\ell_u(\phi') = \ell_u(\phi)$. By (3.4) and the monotonicity of ℓ_v we have

$$\ell_v(\phi) \leq T_e(\ell_u(\phi)) = T_e(\ell_u(\phi')) = \ell_v(\phi') \leq \ell_v(\phi).$$

Thus, we have equality, which means that $e \in E'_\phi$.

$(v) \Rightarrow (iii)$: For $e \in E'_\phi$ we have by Lemma 3.1 (iii) that $F_e^+(\ell_u(\phi)) = F_e^-(T_e(\ell_u(\phi))) = F_e^-(\ell_v(\phi))$. For $e \notin E'_\phi$, let $\phi_0 \in [0, \phi)$ be minimal with $F_e^+(\ell_u(\phi_0)) = F_e^+(\ell_u(\phi))$, which exists due to the contraposition of (v). If $\phi_0 > 0$ we have $F_e^+(\ell_u(\phi_0) - \varepsilon) < F_e^+(\ell_u(\phi_0))$ for all $\varepsilon > 0$, and therefore, we know by (v) that e was active for ϕ_0. It follows from the observation above, from the monotonicity of F_e^- and ℓ_v, as well as from Lemma 3.1 (iii), that

$$F_e^+(\ell_u(\phi)) = F_e^+(\ell_u(\phi_0)) = F_e^-(\ell_v(\phi_0)) \leq F_e^-(\ell_v(\phi)) \leq F_e^-(T_e(\ell_u(\phi))) = F_e^+(\ell_u(\phi)).$$

For $\phi_0 = 0$ we have

$$0 \leq F_e^-(\ell_v(\phi)) \leq F_e^-(T_e(\ell_u(\phi))) = F_e^+(\ell_u(\phi)) = F_e^+(\ell_u(\phi_0)) = 0.$$

In both cases it holds that $F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi))$, which shows (iii). □

Lemma 3.4 (cf. Proposition 2 in [17]). *Given a Nash flow over time the following holds for all particles ϕ:*

(i) $E_\phi^* \subseteq E_\phi'$.

(ii) $E_\phi' = \{\, e = uv \mid \ell_v(\phi) \geq \ell_u(\phi) + \tau_e \,\}$.

(iii) $E_\phi^* = \{\, e = uv \mid \ell_v(\phi) > \ell_u(\phi) + \tau_e \,\}$.

Proof. By Lemma 3.1 (i) we have $e \in E_\phi^* \quad\Leftrightarrow\quad z_e(\ell_u(\phi) + \tau_e) > 0$.

(i) Assume we are given an arc $e \in E_\phi^*$. Either $F_e^+(\ell_u(\phi) - \varepsilon) < F_e^+(\ell_u(\phi))$ for all $\varepsilon > 0$, then $e \in E_\phi'$ by Lemma 3.3 (v), or there exists a $\varphi < \phi$, such that $F_e^+(\ell_u(\varphi)) = F_e^+(\ell_u(\phi))$ and $z_e(\ell_u(\varphi) + \tau_e) > 0$. Using Lemma 3.1 (iv), Equation (3.4) and the monotonicity of ℓ_v we obtain

$$\ell_v(\phi) \leq T_e(\ell_u(\phi)) = T_e(\ell_u(\varphi)) = \ell_v(\varphi) \leq \ell_v(\phi).$$

Thus, we have equality, which means that $e \in E_\phi'$.

(ii) From $e \in E_\phi'$ it follows that $\ell_v(\phi) = \ell_u(\phi) + \tau_e + q_e(\ell_u(\phi)) \geq \ell_u(\phi) + \tau_e$. The reverse inclusion follows, since $q_e(\ell_u(\phi)) > 0$ implies $e \in E_\phi^*$ and by (i) we obtain that $e \in E_\phi'$. For $q_e(\ell_u(\phi)) = 0$ we have

$$\ell_v(\phi) \leq T_e(\ell_u(\phi)) = \ell_u(\phi) + \tau_e \leq \ell_v(\phi),$$

and therefore equality, which shows that e is active for ϕ.

(iii) From $e \in E_\phi^*$ it follows by (i) that e is active, and therefore

$$\ell_v(\phi) = \ell_u(\phi) + \tau_e + q_e(\ell_u(\phi)) > \ell_u(\phi) + \tau_e.$$

The reverse inclusion follows, since

$$\ell_u(\phi) + \tau_e < \ell_v(\phi) \leq T_e(\ell_u(\phi)) = \ell_u(\phi) + \tau_e + q_e(\ell_u(\phi)),$$

and thus, $q_e(\ell_u(\phi)) > 0$, which implies $e \in E_\phi^*$. □

Lemma 3.9. *The α-extension forms a feasible flow over time and the extended ℓ-labels coincide with the earliest arrival times, i.e., they satisfy* (3.4) *for all* $\varphi \in [\phi, \phi+\alpha)$.

Proof. In order to prove that the α-extension forms a flow over time we have to show that the flow conservation is satisfied at every $v \in V \backslash \{t\}$, which is true because for all $\theta \in [\ell_v(\phi), \ell_v(\phi+\alpha))$ it holds that

$$\sum_{e\in\delta_v^+} f_e^+(\theta) - \sum_{e\in\delta_v^-} f_e^-(\theta) = \sum_{e\in\delta_v^+} \frac{x'_e}{\ell'_v} - \sum_{e\in\delta_v^-} \frac{x'_e}{\ell'_v} = \begin{cases} 0 & \text{if } v \in V\setminus\{s,t\} \\ r & \text{if } v = s. \end{cases}$$

For $\theta > \ell_v(\phi+\alpha)$ all functions as well as the network inflow rate are 0, and therefore, the flow conservation holds as well.

Next, we show that the outflow rates obey (3.3). For $e \in E'_\phi$ with $x'_e > 0$ and $\theta \in [\ell_v(\phi), \ell_v(\phi+\alpha))$ we have by (TF3) that

$$\begin{aligned} f_e^-(\theta) = \frac{x'_e}{\ell'_v} &= \begin{cases} \nu_e & \text{if } e \in E^*_\phi, \\ \min\left\{\frac{x'_e}{\ell'_u}, \nu_e\right\} & \text{if } e \in E'_\phi \setminus E^*_\phi, \end{cases} \\ &= \begin{cases} \nu_e & \text{if } q_e(\theta-\tau_e) > 0, \\ \min\{f_e^+(\theta-\tau_e), \nu_e\} & \text{if } q_e(\theta-\tau_e) = 0. \end{cases} \end{aligned}$$

The last equation follows since $e \in E^*_\phi$ implies that $q_e(\ell_u(\phi)) > 0$ and by (3.9) this is also true during the complete thin flow phase. For arcs $e \in E'_\phi \setminus E^*_\phi$ a queue builds up within the phase if, and only if, $\nu_e < \frac{x'_e}{\ell'_u}$ since we have by Lemma 3.1 (vii) that $q'_e(\ell_u(\phi)) = \frac{f_e^+(\ell_u(\phi))}{\nu_e} - 1 = \frac{x'_e}{\ell'_u\cdot\nu_e} - 1$. Furthermore, for active arcs without queues during the thin flow phase we have $[\ell_u(\phi), \ell_u(\phi+\alpha)) = [\ell_v(\phi)-\tau_e, \ell_v(\phi+\alpha)-\tau_e)$. For $x'_e = 0$ (including $e \notin E'_\phi$) we have $f_e^-(\theta) = 0 = f_e^+(\theta-\tau_e)$ and $q_e(\theta-\tau_e) = 0$ for all $\theta \in [\ell_v(\phi), \ell_v(\phi+\alpha))$. Combining all this shows that (3.3) holds for the new phase.

For the second part we show that both equations of (3.4) hold for the extended earliest arrival times. Given an arc $e = uv \in E$, we distinguish between three cases:

Case 1: $e \in E\backslash E'_\phi$.
Since α satisfies Equation (3.10) this equation is also satisfied for all $\xi \in [0,\alpha)$, and hence,

$$\ell_v(\phi+\xi) = \ell_v(\phi) + \xi\cdot\ell'_v \overset{(3.10)}{\leq} \ell_u(\phi) + \xi\cdot\ell'_u + \tau_e \leq T_e(\ell_u(\phi)+\xi\cdot\ell'_u) = T_e(\ell_u(\phi+\xi)).$$

Case 2: $e \in E'_\phi\backslash E^*_\phi$ and $\ell'_u \geq \frac{x'_e}{\nu_e}$.
Since e is active we have $\ell_v(\phi) = T_e(\ell_u(\phi)) = \ell_u(\phi)+\tau_e$ and (TF2) implies $\ell'_v \leq \ell'_u$. There is no queue building up since $f_e^+(\ell_u(\phi+\xi)) = \frac{x'_e}{\ell'_u} \leq \nu_e$, which means $z_e(\ell_u(\phi+\xi)+\tau_e) = 0$ for all $\xi \in [0,\alpha)$. Combining these yields

$$\ell_v(\phi+\xi) = \ell_v(\phi) + \xi\cdot\ell'_v \overset{\text{(TF2)}}{\leq} \ell_u(\phi) + \tau_e + \xi\cdot\ell'_u = \ell_u(\phi+\xi) + \tau_e = T_e(\ell_u(\phi+\xi)).$$

Case 3: $e \in E^*_\phi$ or ($e \in E'_\phi$ and $\ell'_u < \frac{x'_e}{\nu_e}$).
Again, e is active, which means $\ell_v(\phi) = T_e(\ell_u(\phi)) = \ell_u(\phi)+\tau_e+q_e(\ell_u(\phi))$, and additionally, $e \in E^*_\phi$ or $\frac{x'_e}{\ell'_u} > \nu_e$, together with the thin flow condition (TF2), implies that $\ell'_v \leq \frac{x'_e}{\nu_e}$. Lemma 3.1 (vii) implies $q'_e(\ell_u(\phi)) = \frac{f_e^+(\ell_u(\phi))}{\nu_e} - 1 = \frac{x'_e}{\ell'_u\cdot\nu_e} - 1$ since $f_e^+(\ell_u(\phi)) = \frac{x'_e}{\ell'_u} > \nu_e$ in the case of $e \notin E^*_\phi$. By rearranging we obtain $\frac{x'_e}{\nu_e} = q'_e(\ell_u(\phi))\cdot\ell'_u + \ell'_u$. Hence, for all $\xi \in [0,\alpha)$ we obtain with (TF2) that

$$\begin{aligned}
\ell_v(\phi+\xi) &= \ell_v(\phi) + \xi \cdot \ell'_v \\
&\leq \ell_v(\phi) + \xi \cdot \frac{x'_e}{\nu_e} \\
&= \ell_u(\phi) + \tau_e + q_e(\ell_u(\phi)) + \xi \cdot (q'_e(\ell_u(\phi)) \cdot \ell'_u + \ell'_u) \\
&= \ell_u(\phi+\xi) + \tau_e + q_e(\ell_u(\phi) + \xi \cdot \ell'_u) \\
&= T_e(\ell_u(\phi+\xi)).
\end{aligned}$$

This shows that there is no arc with an exit time earlier than the earliest arrival time, and therefore, the left hand side of the second equation in (3.4) is always smaller or equal to the right hand side. It remains to show that the equation holds with equality. For every node $v \in V \setminus \{ s \}$ there is at least one arc $e \in E'$ with $\ell'_v = \rho_e(\ell'_u, x'_e)$ due to (TF2). No matter if this arc belongs to Case 2 or Case 3 the corresponding equation holds with equality, which shows for all $\xi \in [0, \alpha)$ that

$$\ell_v(\phi+\xi) = \min_{e=uv\in E} T_e(\ell_u(\phi+\xi)).$$

This completes the proof. $\square$

4 Time-Dependent Networks

In the base model we assume that the network is constant and does not change over time. In real-world traffic, however, temporary changes of the infrastructure are omnipresent. For example, construction works can cause lane or even road closures for some time duration and school zones reduce the speed limit depending on the time of the day. Surely, the demand, i.e., the network inflow rate, changes drastically between night time and rush hour. Going even further, it is possible to consider traffic lights as time-dependent capacities. We are interested in studying these time-dependent networks because they are capable of modeling more general traffic scenarios than static networks.

In this chapter we will extend the base model by two features: Time-dependent capacities (including the network inflow rate) and time-dependent transit times. As they attach to different parts of the base model, we will consider an extension including both features at the same time. The work in this chapter was developed in collaboration with Julian Steger, who presented the results of time-dependent capacities in his Master's thesis [93], and Hoang Minh Pham, who wrote his Bachelor's thesis on Nash flows over time in networks with time-dependent transit times [42].

4.1 Arc Dynamics

Similar to the base model we consider a directed graph $G = (V, E)$ with a source s and a sink t such that each node is reachable by s. However, this time each arc e is equipped with a time-dependent capacity $\nu_e \colon [0, \infty) \to (0, \infty)$ and a time-dependent speed limit $\lambda_e \colon [0, \infty) \to (0, \infty)$, is inversely proportional to the transit time. In addition, we consider a time-dependent network inflow rate $r \colon [0, \infty) \to [0, \infty)$. We assume that the amount of flow an arc can support is unbounded and that the network inflow is unbounded as well, i.e., for all $e \in E$ we require that

$$\int_0^\theta \nu_e(\xi)\,\mathrm{d}\xi \to \infty, \qquad \int_0^\theta \lambda_e(\xi)\,\mathrm{d}\xi \to \infty \quad \text{and} \quad \int_0^\theta r(\xi)\,\mathrm{d}\xi \to \infty \qquad \text{for } \theta \to \infty.$$

Later on, in order to be able to construct Nash flows over time, we will additionally assume that all these functions are right-constant.

Time-dependent speed limits. Let us focus on the transit times first. We have to be careful how to model the transit time changes, since we do not want to lose the following two properties of the base model:

(i) We want FIFO on arcs, which leads to FIFO on the network for Nash flows over time.

(ii) Particles should never have the incentive to wait on a node.

In other words, we cannot simply allow piecewise-constant transit times, since this could lead to the following case: If the transit time of an arc is high at the beginning and but gets reduced to a lower value at some later point in time, then particles might overtake other particles on that arc. Thus, particles might arrive earlier at the sink if they wait right in front of the arc until its transit time drops.

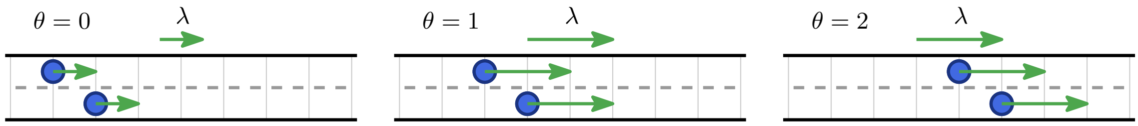


Figure 4.1: Consider a road segment with a time-dependent speed limit that is low in the time interval $[0, 1)$ and large afterwards. All vehicles, independent of their position, first traverse the link slowly and immediately speed up to the new speed limit at time 1.

Hence, we let the speed limit change over time instead. In order to keep the number of parameters of the network as small as possible we assume that the lengths of all arcs equal 1 and, instead of a transit time, we equip every arc $e \in E$ with a time-dependent speed limit $\lambda_e\colon [0, \infty) \to (0, \infty)$. Thus, a particle might traverse the first part of an arc at a different speed than the remaining distance if the maximal speed changes midway; see Figure 4.1.

Transit times. Note that we assume the point queue of an arc to always be right before the exit of the arc, which has not mattered much for the base model, but is important now. Hence, a particle entering arc e at time θ immediately traverses the arc of length 1 with a time-dependent speed of λ_e. The **transit time** $\tau\colon [0, \infty) \to [0, \infty)$ is therefore given by

$$\tau_e(\theta) := \min\left\{ \tau \geq 0 \;\middle|\; \int_\theta^{\theta+\tau} \lambda_e(\xi)\,\mathrm{d}\xi = 1 \right\}.$$

Since we required $\int_0^\theta \lambda_e(\xi)\,\mathrm{d}\xi$ to be unbounded for $\theta \to \infty$, we always have a finite transit time. For an illustrative example see Figure 4.2.

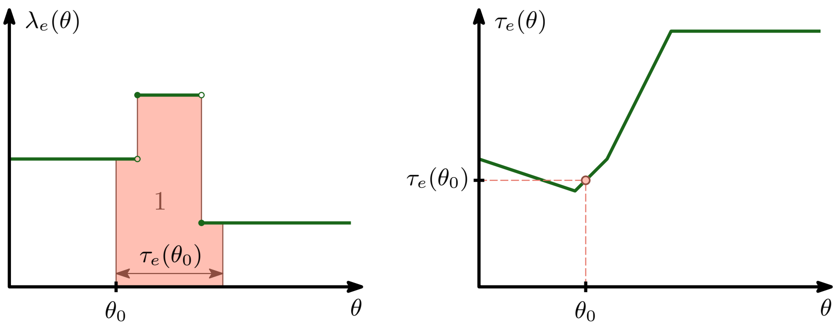


Figure 4.2: From speed limits (*left side*) to transit times (*right side*). The transit time $\tau_e(\theta)$ denotes the time a particle needs to traverse the arc when entering at time θ. We normalize the speed limits by assuming that all arcs have length 1, and hence, the transit time $\tau_e(\theta)$ equals the length of an interval starting at θ such that the area under the speed limit graph within this interval is 1.

The following lemma shows some basic properties of the transit time functions.

Lemma 4.1. *For all $e \in E$ and almost all $\theta \in [0, \infty)$ we have:*

(i) The function $\theta \mapsto \theta + \tau_e(\theta)$ is strictly increasing.

(ii) The function τ_e is continuous and almost everywhere differentiable.

(iii) For almost all $\theta \in [0, \infty)$ we have

$$1 + \tau_e'(\theta) = \frac{\lambda_e(\theta)}{\lambda_e(\theta + \tau_e(\theta))}.$$

Proof.

(i) Consider two points in time $\theta_1 < \theta_2$, then $\tau := \theta_1 - \theta_2 + \tau_e(\theta_1)$ is strictly smaller than $\tau_e(\theta_2)$ since

$$\int_{\theta_2}^{\theta_2+\tau} \lambda_e(\xi)\,\mathrm{d}\xi = \int_{\theta_2}^{\theta_1+\tau_e(\theta_1)} \lambda_e(\xi)\,\mathrm{d}\xi < \int_{\theta_1}^{\theta_1+\tau_e(\theta_1)} \lambda_e(\xi)\,\mathrm{d}\xi = 1,$$

where the strict inequality holds, since λ_e is always strictly positive. The last equality follows by the definition of $\tau_e(\theta_1)$. Hence, with the definition of $\tau_e(\theta_2)$ we have $\theta_1 + \tau_e(\theta_1) < \theta_2 + \tau_e(\theta_2)$.

(ii) Since $\theta \mapsto \theta + \tau_e(\theta)$ is monotone, Lebesgue's theorem for the differentiability of monotone functions (Theorem 2.5 on page 19) implies that it is almost everywhere differentiable. The same is then true for τ_e. The continuity follows directly from the definition since λ_e is always strictly positive.

(iii) By the definition of $\tau_e(\theta)$ we have

$$\int_0^{\theta+\tau_e(\theta)} \lambda_e(\xi)\,\mathrm{d}\xi - \int_0^{\theta} \lambda_e(\xi)\,\mathrm{d}\xi = 1.$$

Taking the derivatives of both sides and using Lebesgue's differentiation theorem (Theorem 2.4 on page 18) together with the chain rule, we obtain

$$\lambda_e(\theta + \tau_e(\theta)) \cdot (1 + \tau_e'(\theta)) - \lambda_e(\theta) = 0.$$

Since λ_e is always strictly positive, we get

$$1 + \tau_e'(\theta) = \frac{\lambda_e(\theta)}{\lambda_e(\theta + \tau_e(\theta))}.$$

□

Speed ratios. The ratio in Lemma 4.1 (iii) will be important to measure the outflow of an arc depending on the inflow. We call $\gamma_e \colon [0,\infty) \to [0,\infty)$ the **speed ratio** of e and it is defined by

$$\gamma_e(\theta) := \frac{\lambda_e(\theta)}{\lambda_e(\theta + \tau_e(\theta))} = 1 + \tau_e'(\theta).$$

Figuratively speaking, this ratio describes how much the flow rate changes under different speed limits. If, for example, $\gamma_e(\theta) = 2$, as depicted in Figure 4.3, this means that the speed limit was twice as high when the particle entered the arc as it is at the moment the particle enters the queue. In this case the flow rate is halved on its way, since the same amount of flow that entered within one time unit, needs two time units to leave it. With the same intuition the flow rate is increased whenever $\gamma_e(\theta) < 1$. Note that we normally picture the flow rate by the width of the flow in the figures. But

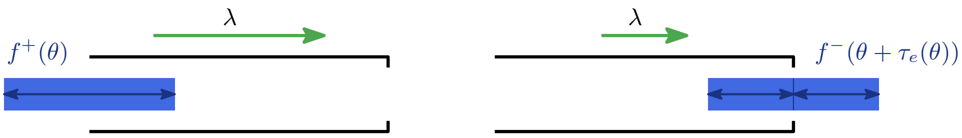


Figure 4.3: An illustration of how the flow rate changes depending on the speed limits. *On the left:* As the speed limit λ is high, the flow volume entering the arc per time unit is represented by the area of the long rectangle. *On the right:* The speed limit is halved, and therefore, the same amount of flow needs twice as much time to leave the arc (or enter the queue if there is one). Hence, if there is no queue, the outflow rate at time $\tau + \tau_e(\theta)$ is only half the size of the inflow rate at time θ.

for time-dependent networks this is not accurate anymore as the transit speed can vary. Hence, in this chapter the flow rates are given by the width of the flow multiplied by the current speed limit.

Flows over time. A **flow over time** in a time-dependent network is, as before, a family of locally integrable and bounded functions $f = (f_e^+, f_e^-)_{e \in E}$ that **conserves flow on all arcs** e:

$$F_e^-(\theta + \tau_e(\theta)) \leq F_e^+(\theta) \qquad \text{for all } \theta \in [0, \infty], \tag{4.1}$$

and **conserves flow at every node** $v \in V \setminus \{t\}$ for almost all $\theta \in [0, \infty)$:

$$\sum_{e \in \delta_v^+} f_e^+(\theta) - \sum_{e \in \delta_v^-} f_e^-(\theta) = \begin{cases} 0 & \text{if } v \in V \setminus \{t\}, \\ r(\theta) & \text{if } v = s. \end{cases} \tag{4.2}$$

The functions $F_e^+, F_e^- : [0, \infty) \to [0, \infty)$ denote, again, the **cumulative in- and outflow** defined by

$$F_e^+(\theta) := \int_0^\theta f_e^+(\xi) \,\mathrm{d}\xi \qquad \text{and} \qquad F_e^-(\theta) := \int_0^\theta f_e^-(\xi) \,\mathrm{d}\xi.$$

Queues. A particle entering an arc e at time θ reaches the head of the arc at time $\theta + \tau_e(\theta)$ where it lines up at the point queue. Thereby, the **queue size** $z_e : [0, \infty) \to [0, \infty)$ at time $\theta + \tau_e(\theta)$ is defined by

$$z_e(\theta + \tau_e(\theta)) := F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta)).$$

Feasibility. We call a flow over time in a time-dependent network **feasible** if we have for almost all $\theta \in [0, \infty)$ that

$$f_e^-(\theta + \tau_e(\theta)) = \begin{cases} \nu_e(\theta + \tau_e(\theta)) & \text{if } z_e(\theta + \tau_e(\theta)) > 0, \\ \min\left\{ \frac{f_e^+(\theta)}{\gamma_e(\theta)}, \nu_e(\theta + \tau_e(\theta)) \right\} & \text{else,} \end{cases} \tag{4.3}$$

and $f_e^-(\theta) = 0$ for almost all $\theta < \tau_e(0)$.
Note that the outflow rate depends on the speed ratio $\gamma_e(\theta)$ if the queue is empty (see Figure 4.3). Otherwise, the particles enter the queue, and therefore, the outflow rate equals the capacity independent of the speed ratio. Furthermore, we observe that every arc with a positive queue always has a positive outflow, since the capacities are required to be strictly positive. And finally, (4.3) implies (4.1), the same way it does in the base model. This can easily be seen by considering the derivatives of the cumulative flows whenever we have an empty queue, i.e., $F_e^-(\theta + \tau_e(\theta)) = F_e^+(\theta)$. By (4.3) we have that $f_e^-(\theta + \tau_e(\theta)) \cdot (1 + \tau_e'(\theta)) \leq f_e^+(\theta)$. Hence, Conditions (4.2) and (4.3) are sufficient for a family of functions $f = (f_e^+, f_e^-)_{e \in E}$ to be a feasible flow over time.

Waiting times. The **waiting time** $q_e : [0, \infty) \to [0, \infty)$ of a particle that enters the arc at time θ is now defined by

$$q_e(\theta) := \min\left\{ q \geq 0 \;\middle|\; \int_{\theta + \tau_e(\theta)}^{\theta + \tau_e(\theta) + q} \nu_e(\xi) \,\mathrm{d}\xi = z_e(\theta + \tau_e(\theta)) \right\}.$$

As we required $\int_0^\theta \nu_e(\xi) \,\mathrm{d}\xi$ to be unbounded for $\theta \to \infty$ the set on the right side is never empty. Hence, $q_e(\theta)$ is well-defined and has a finite value. In addition, q_e is continuous since ν_e is always strictly positive.

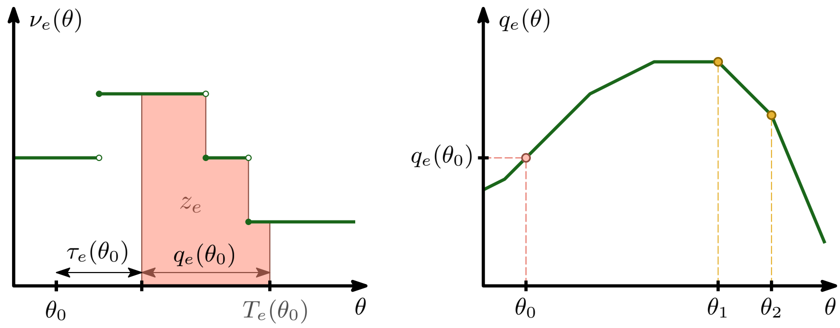


Figure 4.4: Waiting times for time-dependent capacities. The waiting time of a particle θ_0 (*right side*) is given by the length of the interval starting at $\theta_0 + \tau_e(\theta_0)$ such that the area underneath the capacity graph equals the queue size at time $\theta_0 + \tau_e(\theta_0)$ (*left side*). Note that the right boundary of the interval equals the exit time $T_e(\theta_0)$. Clearly, the waiting time does not only depend on the capacity but also on the inflow rate and the transit times (speed limits). For example, if the capacity and the speed limit are constant but the inflow rate is 0, the waiting time will decrease with a slope of 1 as pictured *on the right side* within $[\theta_1, \theta_2]$. But it can decline even quicker if, in addition, the transit time is increasing at time θ, i.e., if $\gamma_e(\theta) > 1$.

Exit times. The **exit time** $T_e\colon [0,\infty) \to [0,\infty)$ denotes the time at which the particles that have entered the arc at time θ finally leave the queue. Hence, we define

$$T_e(\theta) := \theta + \tau_e(\theta) + q_e(\theta).$$

In Figure 4.4 we display an illustrative example for the definition of waiting and exit times. With these definitions we can show the following lemma.

Lemma 4.2. *For a feasible flow over time f it holds for all $e \in E$, $v \in V$ and $\theta \in [0,\infty)$ that:*

(i) $q_e(\theta) > 0 \quad \Leftrightarrow \quad z_e(\theta + \tau_e(\theta)) > 0.$

(ii) $z_e(\theta + \tau_e(\theta) + \xi) > 0$ *for all* $\xi \in [0, q_e(\theta))$.

(iii) $F_e^+(\theta) = F_e^-(T_e(\theta))$.

(iv) For $\theta_1 < \theta_2$ *with* $F_e^+(\theta_2) - F_e^+(\theta_1) = 0$ *and* $z_e(\theta_2 + \tau_e(\theta_2)) > 0$ *we have* $T_e(\theta_1) = T_e(\theta_2)$.

(v) The functions T_e are monotonically increasing.

(vi) The functions q_e and T_e are continuous and almost everywhere differentiable.

(vii) For almost all $\theta \in [0,\infty)$ we have

$$T_e'(\theta) = \begin{cases} \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} & \text{if } q_e(\theta) > 0, \\ \max\left\{ \gamma_e(\theta), \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} \right\} & \text{else.} \end{cases}$$

The proof is very similar to the proof of Lemma 3.1 but, due to the time-dependent speed limits and capacities, some more technical calculations are involved. The proof with all details can be found in the appendix on page 64.

4.2 Nash Flows Over Time

As in the base model a dynamic equilibrium is a feasible flow over time, where particles only use current shortest paths from s to t. Note that we still assume a game with full information. Consequently, all particles know all speed limit and capacity functions in advance and choose their routes accordingly. Again, we start by defining the earliest arrival times and the current shortest paths network, now for the extended model.

Earliest arrival times. The **earliest arrival time functions** $\ell_v \colon \mathbb{R}_{\geq 0} \to [0, \infty)$ are defined by

$$\ell_v(\phi) := \begin{cases} \min \left\{ \theta \geq 0 \,\middle|\, \int_0^\theta r(\xi)\,\mathrm{d}\xi = \phi \right\} & \text{for } v = s, \\ \min\limits_{e = uv \in \delta_v^-} T_e(\ell_u(\phi)) & \text{else.} \end{cases} \tag{4.4}$$

They are well-defined as we have finite speed limits, and hence, $T_e(\ell_u(\theta))$ is strictly larger than $\ell_u(\theta)$. Note that for all $v \in V$ the earliest arrival time function ℓ_v is monotonically increasing, continuous and almost everywhere differentiable. This holds directly for ℓ_s and for $v \neq s$ it follows inductively, since these properties are preserved by the composition $T_e \circ \ell_u$ and by the minimum of finitely many functions.

Active arcs, resetting arcs and current shortest paths networks. As before we denote the **active arcs** and the **resetting arcs** for a particle ϕ by

$$E'_\phi := \{\, e = uv \in E \mid \ell_v(\phi) = T_e(\ell_u(\phi)) \,\} \quad \text{and} \quad E^*_\phi := \{\, e = uv \in E \mid q_e(\ell_u(\phi)) > 0 \,\}$$

and the **current shortest paths network** by $G'_\phi = (V, E'_\phi)$. Note that G'_ϕ is acyclic and that every node is reachable by s within this graph.

Dynamic equilibria. Nash flows over time in time-dependent networks are now defined in the exact same way as they were defined in the base model.

Definition 4.3 (Nash flow over time in a time-dependent network). We call a feasible flow over time f a **Nash flow over time** if the following **Nash flow condition** holds:

$$f_e^+(\theta) > 0 \quad \Rightarrow \quad \theta \in \ell_u(\Phi_e) \qquad \text{for all arcs } e = uv \in E \text{ and almost all } \theta \in [0, \infty), \tag{N}$$

where $\Phi_e := \{\, \phi \in \mathbb{R}_{\geq 0} \mid e \in E'_\phi \,\}$ is the set of flow particles for which arc e is active.

And again, they have the same characterization.

Lemma 4.4. *Let f be a feasible flow over time. Then the following statements are equivalent:*

(i) f is a Nash flow over time.

(ii) For all arcs $e = uv \in E$ and all particles $\phi \in \mathbb{R}_{\geq 0}$ we have $F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi))$.

Since the exit and the earliest arrival times have the same properties in time-dependent networks as in the base model, this lemma follows with the exact same proof as the corresponding Lemma 3.3 in the base model (the proof can be found on page 47).
The following lemma also transfers one-to-one from the base model.

Lemma 4.5. *Given a Nash flow over time the following holds for all particles ϕ:*

(i) $E^*_\phi \subseteq E'_\phi$.

(ii) $E'_\phi = \{ e = uv \mid \ell_v(\phi) \geq \ell_u(\phi) + \tau_e(\theta) \}$.

(iii) $E^*_\phi = \{ e = uv \mid \ell_v(\phi) > \ell_u(\phi) + \tau_e(\theta) \}$.

By replacing τ_e by $\tau_e(\ell_u(\theta))$ the proof for Lemma 3.4 on page 49 also holds for time-dependent networks.

Underlying static flows. Again, we define the **underlying static flow** for every $\phi \in \mathbb{R}_{\geq 0}$ by

$$x_e(\phi) := F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi)) \qquad \text{for all } e = uv \in E.$$

By the definition of ℓ_s and the integration of (4.2) we have $\int_0^{\ell_s(\phi)} r(\xi)\,\mathrm{d}\xi = \phi$, and hence, $x_e(\phi)$ is a static s-t-flow of value ϕ, whereas the derivatives $(x'_e(\phi))_{e\in E}$ form a static s-t-flow of value 1.

4.3 Thin Flows

In this section we want to transfer the concept of thin flows with resetting to time-dependent networks. Since thin flows should characterize the derivatives of a Nash flow over time, we have to consider the impact on time-dependent capacities and speed limits on these derivatives.
Consider an acyclic network $G' = (V, E')$ with a source s and a sink t, such that every node is reachable by s. Each arc is equipped with a capacity $\nu_e > 0$ and a speed ratio $\gamma_e > 0$. Furthermore, we have a network inflow rate of $r > 0$ and an arc set $E^* \subseteq E'$. We obtain the following definition.

Definition 4.6 (Thin flow with resetting in a time-dependent network).
A static s-t flow $(x'_e)_{e\in E}$ of value 1 together with a node labeling $(\ell'_v)_{v\in V}$ is a **thin flow with resetting** on E^* if:

$$\ell'_s = \frac{1}{r} \tag{TF1}$$

$$\ell'_v = \min_{e=uv\in E'} \rho_e(\ell'_u, x'_e) \qquad \text{for all } v \in V \setminus \{ s \}, \tag{TF2}$$

$$\ell'_v = \rho_e(\ell'_u, x'_e) \qquad \text{for all } e = uv \in E' \text{ with } x'_e > 0, \tag{TF3}$$

$$\text{where} \qquad \rho_e(\ell'_u, x'_e) := \begin{cases} \frac{x'_e}{\nu_e} & \text{if } e = uv \in E^*, \\ \max\left\{ \gamma_e \cdot \ell'_u, \frac{x'_e}{\nu_e} \right\} & \text{if } e = uv \in E' \setminus E^*. \end{cases}$$

The derivatives of a Nash flow over time in time-dependent networks do indeed form a thin flow with resetting as the following theorem shows.

Theorem 4.7.
For almost all $\phi \in \mathbb{R}_{\geq 0}$ the derivatives $(x'_e(\phi))_{e\in E'_\phi}$ and $(\ell'_v(\phi))_{v\in V}$ of a Nash flow over time $f = (f_e^+, f_e^-)_{e\in E}$ form a thin flow with resetting on E^*_ϕ in the current shortest paths network $G'_\phi = (V, E'_\phi)$ with network inflow rate $r(\ell_s(\phi))$ as well as capacities $\nu_e(\ell_v(\phi))$ and speed ratios $\gamma_e(\ell_u(\phi))$ for each arc $e = uv \in E$.

Proof. Let $\phi \in \mathbb{R}_{\geq 0}$ be a particle such that for all arcs $e = uv \in E$ the derivatives of x_e, ℓ_u, $T_e \circ \ell_u$ and τ_e exist and $x'_e(\phi) = f^+_e(\ell_u(\phi)) \cdot \ell'_u(\phi) = f^-_e(\ell_v(\phi)) \cdot \ell'_v(\phi)$ as well as $1 + \tau'_e(\ell_u(\phi)) = \gamma_e(\ell_u(\phi))$. This is given for almost all ϕ.
By (4.4) we have $\int_0^{\ell_s(\phi)} r(\xi)\,\mathrm{d}\xi = \phi$ and taking the derivative by applying the chain rule, yields $r(\ell_s(\phi)) \cdot \ell'_s(\phi) = 1$, which shows (TF1).
Taking the derivative of (4.4) at time $\ell_u(\phi)$ by using the differentiation rule for a minimum (Lemma 2.3 on page 17) yields

$$\ell'_v(\phi) = \min_{e=uv\in E'} T'_e(\ell_u(\phi)) \cdot \ell'_u(\phi).$$

By using Lemma 4.2 (vii) we obtain

$$T'_e(\ell_u(\phi)) \cdot \ell'_u(\phi) = \left\{ \begin{array}{ll} \frac{f^+_e(\ell_u(\phi))}{\nu_e(T_e(\ell_u(\phi)))} \cdot \ell'_u(\phi) & \text{if } q_e(\ell_u(\phi)) > 0, \\ \max\left\{ \gamma_e(\ell_u(\phi)), \frac{f^+_e(\ell_u(\phi))}{\nu_e(T_e(\ell_u(\phi)))} \right\} \cdot \ell'_u(\phi) & \text{else,} \end{array} \right\} = \rho_e(\ell'_u(\phi), x'_e(\phi)),$$

which shows (TF2).
Finally, in the case of $f^-_e(\ell_v(\phi)) \cdot \ell'_v(\phi) = x'_e(\phi) > 0$ we have by (4.3) that

$$\begin{aligned} \ell'_v(\phi) = \frac{x'_e(\phi)}{f^-_e(\ell_v(\phi))} &= \begin{cases} \dfrac{x'_e(\phi)}{\min\left\{ \frac{f^+_e(\ell_u(\phi))}{\gamma_e(\ell_u(\phi))}, \nu_e(\ell_v(\phi)) \right\}} & \text{if } q_e(\ell_u(\phi)) = 0, \\ \dfrac{x'_e(\phi)}{\nu_e(\ell_v(\phi))} & \text{else,} \end{cases} \\ &= \left\{ \begin{array}{ll} \max\left\{ \gamma_e(\ell_u(\phi)) \cdot \ell'_u(\phi), \frac{x'_e(\phi)}{\nu_e(\ell_v(\phi))} \right\} & \text{if } e \in E'_\phi \setminus E^*_\phi, \\ \frac{x'_e(\phi)}{\nu_e(\ell_v(\phi))} & \text{if } e \in E^*_\phi, \end{array} \right\} = \rho_e(\ell'_u(\phi), x'_e(\phi)). \end{aligned}$$

This shows (TF3) and finishes the proof. □

In order to construct Nash flows over time in time-dependent networks in the next section, we have to show that there always exists a thin flow with resetting in this setting.

Theorem 4.8.
Consider an acyclic graph $G' = (V, E')$ with source s, sink t, capacities $\nu_e > 0$, speed ratios $\gamma_e > 0$ and a subset of arcs $E^* \subseteq E'$, as well as a network inflow $r > 0$. Furthermore, suppose that every node is reachable from s. Then there exists a thin flow $((x'_e)_{e\in E}, (\ell'_v)_{v\in V})$ with resetting on E^*.

This proof works exactly as the proof for the existence of thin flows in the base model; see Theorem 3.7 on page 35.

4.4 Constructing Nash Flows Over Time

In the remaining part of this chapter we assume that for all $e \in E$ the functions ν_e and λ_e as well as the network inflow rate function r are right-constant. In order to show the existence of Nash flows over time in time-dependent networks we use the same procedure as in the base model. We start with the empty flow over time and expand it step by step by using a thin flow with resetting.

α-Extension. Given a **restricted Nash flow over time** f on $[0, \phi]$, i.e., a Nash flow over time where only the particles in $[0, \phi]$ are considered, we obtain well-defined earliest arrival times $(\ell_v(\phi))_{v\in V}$ for particle ϕ. Hence, by Lemma 4.5 we can determine the current shortest paths network $G'_\phi = (V, E'_\phi)$ with the resetting arcs E^*_ϕ, the capacities $\nu_e(\ell_v(\phi))$ and speed ratios $\gamma_e(\ell_u(\phi))$ for all arcs $e = uv \in E'$ as well as the network inflow rate $r(\ell_s(\phi))$. By Theorem 4.8 there exists a thin flow $((x'_e)_{e\in E'}, (\ell'_v)_{v\in V})$ on G'_ϕ with resetting on E^*_ϕ. For $e \notin E'_\phi$ we set $x'_e := 0$. As in the base model we extend the ℓ- and x-functions for some $\alpha > 0$ by

$$\ell_v(\phi + \xi) := \ell_v(\phi) + \xi \cdot \ell'_v \quad \text{and} \quad x_e(\phi) := x_e(\phi) + \xi \cdot x'_e \qquad \text{for all } \xi \in [0, \alpha]$$

and the in- and outflow rate functions by

$$f^+_e(\theta) := \frac{x'_e}{\ell'_u} \quad \text{for } \theta \in [\ell_u(\phi), \ell_u(\phi+\alpha)) \qquad \text{and} \qquad f^-_e(\theta) := \frac{x'_e}{\ell'_v} \quad \text{for } \theta \in [\ell_v(\phi), \ell_v(\phi+\alpha)).$$

We call this extended flow over time α-**extension**. Note that $\ell'_u = 0$ means that $[\ell_u(\phi), \ell_u(\phi + \alpha))$ is empty, and the same holds for ℓ'_v.

Feasible extension step size. An α-extension is a restricted Nash flow over time, which we will prove later on, as long as the α stays within reasonable bounds. Similar to the base model we have to ensure that resetting arcs stay resetting and non-active arcs stay non-active for all particles in $[\phi, \phi + \alpha)$. Since the transit times may now vary over time, we have to alter the conditions of the base model:

$$\ell_v(\phi) + \xi \cdot \ell'_v - \ell_u(\phi) - \xi \cdot \ell'_u > \tau_e(\ell_u(\phi) + \xi \cdot \ell'_u)) \quad \text{for every } e \in E^*_\phi \text{ and all } \xi \in [0, \alpha), \tag{4.5}$$

$$\ell_v(\phi) + \xi \cdot \ell'_v - \ell_u(\phi) - \xi \cdot \ell'_u < \tau_e(\ell_u(\phi) + \xi \cdot \ell'_u)) \quad \text{for every } e \in E \setminus E'_\phi \text{ and all } \xi \in [0, \alpha). \tag{4.6}$$

Furthermore, we need to ensure that the capacities of all active arcs and the network inflow rate do not change within the phase:

$$\nu_e(\ell_v(\phi)) = \nu_e(\ell_v(\phi) + \xi \cdot \ell'_v) \quad \text{for every } e \in E'_\phi \text{ and all } \xi \in [0, \alpha). \tag{4.7}$$

$$r(\ell_s(\phi)) = r(\ell_s(\phi) + \xi \cdot \ell'_s) \quad \text{for all } \xi \in [0, \alpha). \tag{4.8}$$

Finally, the speed ratios need to stay constant for all active arcs, i.e.,

$$\gamma_e(\ell_u(\phi)) = \gamma_e(\ell_u(\phi) + \xi \cdot \ell'_u) \quad \text{for every } e \in E'_\phi \text{ and all } \xi \in [0, \alpha). \tag{4.9}$$

We call an $\alpha > 0$ **feasible** if it satisfies (4.5) to (4.9).

Remark 4.9. *The Conditions* (4.5) *to* (4.9) *on α are sufficient to guarantee that the α-extension is a Nash flow over time as we will show in Theorem 4.12. But it is worth noting that it is possible to formulate more complex conditions on α in order to have longer thin flow phases by skipping events that do not change the thin flow. For example, Equation* (4.9) *only needs to hold for arcs that stay active during the phase, i.e., arcs for which* (TF3) *is satisfied. But if we restrict it to those, we need to ensure that active arcs leaving the current shortest path network immediately, also satisfy Equation* (4.6) *for all $\xi \in (0, \alpha)$ as these arcs might become active again within the same phase due to a change of the speed ratio. As these improved conditions are more complicated to state we will only consider the conditions stated above in this thesis. But for an actual implementation there is room for improvement as a reduction in the number of the costly thin flow computations can speed up the algorithm significantly.*

Lemma 4.10. *Given a restricted Nash flow over time f on $[0, \phi]$ then for right-constant capacities and speed limits there always exists a feasible $\alpha > 0$.*

Proof. By Lemma 4.5 we have that $\ell_v(\phi) - \ell_u(\phi) > \tau_e(\phi)$ for all $e \in E^*_\phi$ and $\ell_v(\phi) - \ell_u(\phi) < \tau_e(\phi)$ for all $e \in E \setminus E'_\phi$. Since τ_e is continuous there is an $\alpha_1 > 0$ such that (4.5) and (4.6) are satisfied for all $\xi \in [0, \alpha_1)$. Since ν_e, r and λ_e are right-constant so is γ_e, and hence, there is an $\alpha_2 > 0$ such that (4.7), (4.8) and (4.9) are fulfilled for all $\xi \in [0, \alpha_2)$. Clearly, $\alpha := \min\{\alpha_1, \alpha_2\} > 0$ is feasible. □

For the maximal feasible α we call the interval $[\phi, \phi + \alpha)$ a **thin flow phase**.

Lemma 4.11. *An α-extension is a feasible flow over time and the extended ℓ-labels coincide with the earliest arrival times, i.e., they satisfy Equation* (4.4) *for all $\varphi \in [\phi, \phi + \alpha)$.*

The proof is similar to the proof of the corresponding Lemma 3.9 in the base model, but the time-dependent transit times and capacities make it a bit more involved. It can be found in the appendix on page 66.
The final step is to show that an α-extension is a restricted Nash flow over time on $[0, \phi + \alpha)$ and that we can continue this process up to ∞.

Theorem 4.12.

Given a restricted Nash flow over time $f = (f_e^+, f_e^-)_{e \in E}$ on $[0, \phi)$ in a time-dependent network and a feasible $\alpha > 0$ then the α-extension is a restricted Nash flow over time on $[0, \phi + \alpha)$.

Theorem 4.13.

There exists a Nash flow over time in every time-dependent network with right-constant speed limits, capacities and network inflow rates.

The proofs of both theorems are exactly the same as the proofs for the corresponding Theorems 3.10 and 3.11 on page 38.

Example. An example of a Nash flow over time in a time-dependent network together with the corresponding thin flows is shown in Figure 4.5 on the next page.
As displayed at the top the capacity of arc su drops from 2 to 1 at time 8 and, at the same time, the speed limit of arc vt decreases from $\frac{1}{2}$ to $\frac{1}{6}$. The first event for particle 4 is due to a change of the speed ratio leading to an increase of ℓ'_t. For particle 6, the top path becomes active and is taken by all following flow as particles on arc vt are still slowed down. For particle 8, the speed ratio at arc vt changes back to 1 but, as this arc is inactive, this does not change anything. Particle 12 is the first to experience the reduced capacity on arc su. The corresponding queue of this arc increases until the bottom path becomes active. This happens in two steps: first only the path up to node v becomes active for $\phi = 16$, and finally, the complete path is active from $\phi = 20$ onwards.

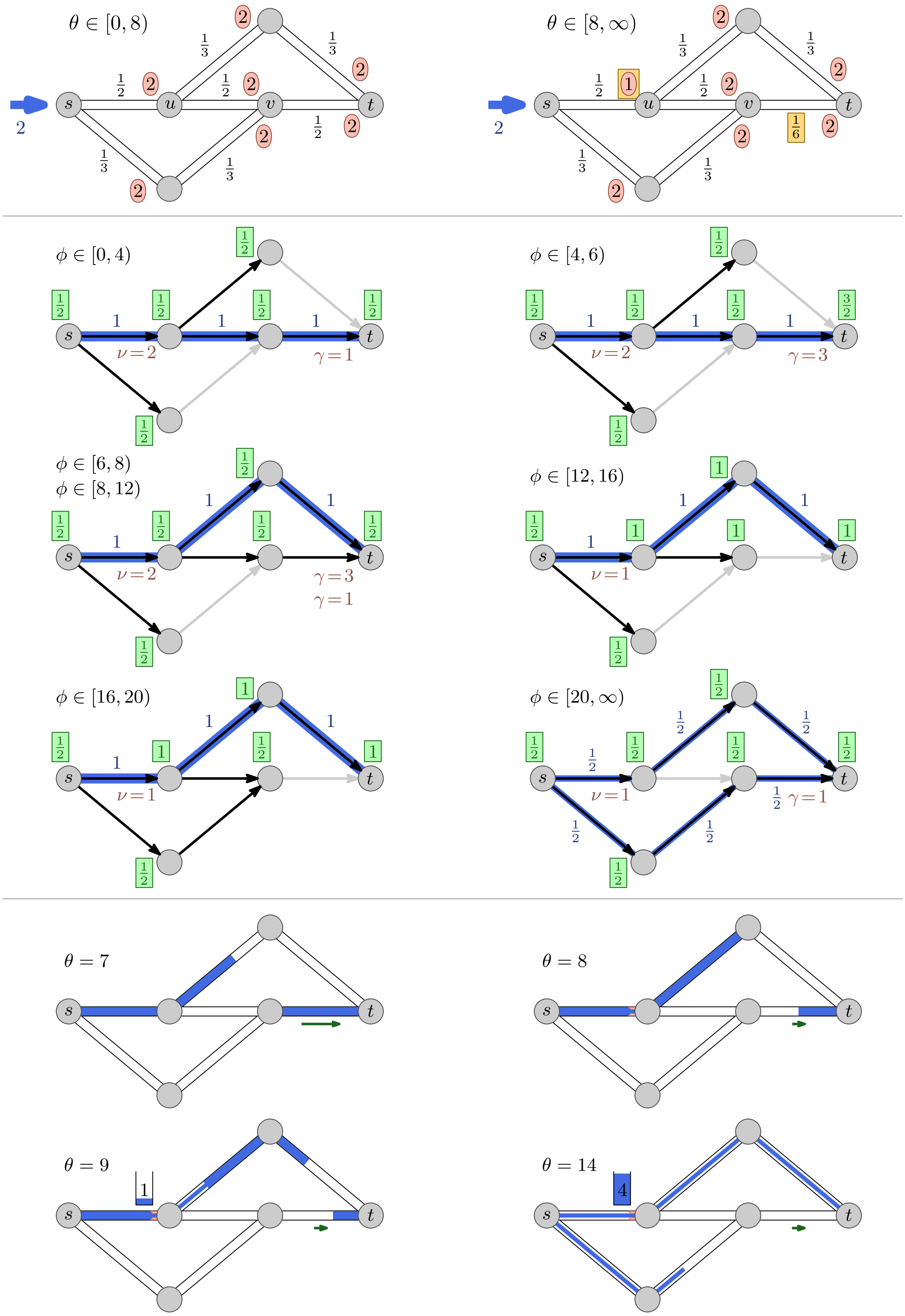


Figure 4.5: A Nash flow over time with corresponding thin flows in a time-dependent network. *On the top:* The network properties before time 8 (*left side*) and after time 8 (*right side*). *In the middle:* There are seven thin flow phases. Note that the third and forth phase (both depicted in the same network) are almost identical and only the speed ratio of arc vt changes, which does not influence the thin flow at all. *At the bottom:* Some key snapshots in time of the resulting Nash flow over time. The current speed limit of arc vt is visualized by the length of the green arrow and, from time 8 onwards, the reduced capacity on arc su is displayed by a red bottle-neck.

4.5 Appendix: Technical Proofs

Lemma 4.2. *For a feasible flow over time f it holds for all $e \in E$, $v \in V$ and $\theta \in [0, \infty)$ that:*

(i) $q_e(\theta) > 0 \quad \Leftrightarrow \quad z_e(\theta + \tau_e(\theta)) > 0.$

(ii) $z_e(\theta + \tau_e(\theta) + \xi) > 0$ *for all* $\xi \in [0, q_e(\theta))$.

(iii) $F_e^+(\theta) = F_e^-(T_e(\theta))$.

(iv) For $\theta_1 < \theta_2$ *with* $F_e^+(\theta_2) - F_e^+(\theta_1) = 0$ *and* $z_e(\theta_2 + \tau_e(\theta_2)) > 0$ *we have* $T_e(\theta_1) = T_e(\theta_2)$.

(v) The functions T_e *are monotonically increasing.*

(vi) The functions q_e *and* T_e *are continuous and almost everywhere differentiable.*

(vii) For almost all $\theta \in [0, \infty)$ *we have*

$$T_e'(\theta) = \begin{cases} \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} & \text{if } q_e(\theta) > 0, \\ \max\left\{ \gamma_e(\theta), \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} \right\} & \text{else.} \end{cases}$$

Proof.

(i) This follows directly from the definition of the waiting time q_e.

(ii) By (4.3) we have that $f_e^-(\xi) \leq \nu_e(\xi)$ almost everywhere. Hence, we have by definition that $q_e(\theta)$ is the minimal value such that

$$\int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+q_e(\theta)} \nu_e(\xi)\,\mathrm{d}\xi = z_e(\theta + \tau_e(\theta)).$$

Thus, we obtain for $\xi \in [0, q_e(\theta))$ that

$$\begin{aligned} F_e^-(\theta + \tau_e(\theta) + \xi) - F_e^-(\theta + \tau_e(\theta)) &= \int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+\xi} f_e^-(\xi)\,\mathrm{d}\xi \\ &\leq \int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+\xi} \nu_e(\xi)\,\mathrm{d}\xi \\ &< z_e(\theta + \tau_e(\theta)) \\ &= F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta)). \end{aligned}$$

Or in short: $F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta) + \xi) > 0$ for $\xi \in [0, q_e(\theta))$. Since F_e^+ is monotonically increasing we obtain for all $\xi \in [0, q_e(\theta))$ that

$$z_e(\theta + \tau_e(\theta) + \xi) = F_e^+(\theta + \xi) - F_e^-(\theta + \tau_e(\theta) + \xi) \geq F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta) + \xi) > 0.$$

(iii) Again by (4.3) together with (ii) we obtain for almost all $\xi \in [\theta + \tau_e(\theta), \theta + \tau_e(\theta) + q_e(\theta))$ that $f_e^-(\xi) = \nu_e(\xi)$. By the definition of q_e we have

$$\begin{aligned}F_e^-(\theta + \tau_e(\theta) + q_e(\theta)) - F_e^-(\theta + \tau_e(\theta)) &= \int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+q_e(\theta)} f_e^-(\xi)\,\mathrm{d}\xi \\ &= \int_{\theta+\tau_e(\theta)}^{\theta+\tau_e(\theta)+q_e(\theta)} \nu_e(\xi)\,\mathrm{d}\xi \\ &= z_e(\theta + \tau_e(\theta)) \\ &= F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta)).\end{aligned}$$

Hence, $F_e^-(T_e(\theta)) = F_e^+(\theta)$.

(iv) Since $F_e^+(\theta_1) = F_e^+(\theta_2)$ we obtain with the monotonicity of F_e^- and Lemma 4.1 (i) that

$$z_e(\xi + \tau_e(\xi)) = F_e^+(\xi) - F_e^-(\xi + \tau_e(\xi)) \geq F_e^+(\theta_2) - F_e^-(\theta_2 + \tau_e(\theta_2)) = z_e(\theta_2 + \tau_e(\theta_2)) > 0,$$

and therefore, (4.3) provides $f_e^-(\xi) = \nu_e(\xi)$ for almost all $\xi \in [\theta_1 + \tau_e(\theta_1), \theta_2 + \tau_e(\theta_2)]$.

Thus, the definition of q_e implies that $q_e(\theta_1)$ equals

$$\begin{aligned}&\min\left\{ q \geq 0 \;\middle|\; \int_{\theta_1+\tau_e(\theta_1)}^{\theta_2+\tau_e(\theta_2)} f_e^-(\xi)\,\mathrm{d}\xi + \int_{\theta_2+\tau_e(\theta_2)}^{\theta_1+\tau_e(\theta_1)+q} \nu_e(\xi)\,\mathrm{d}\xi = F_e^+(\theta_1) - F_e^-(\theta_1 + \tau_e(\theta_1)) \right\} \\ &= \min\left\{ p \geq 0 \;\middle|\; \int_{\theta_2+\tau_e(\theta_2)}^{\theta_2+\tau_e(\theta_2)+p} \nu_e(\xi)\,\mathrm{d}\xi = F_e^+(\theta_2) - F_e^-(\theta_2 + \tau_e(\theta_2)) \right\} + \theta_2 + \tau_e(\theta_2) - \theta_1 - \tau_e(\theta_1) \\ &= q_e(\theta_2) + \theta_2 + \tau_e(\theta_2) - \theta_1 - \tau_e(\theta_1).\end{aligned}$$

Here, the first equation can be obtained by substituting q by $p + \theta_2 + \tau_e(\theta_2) - \theta_1 - \tau_e(\theta_1)$. Note that the condition $p \geq 0$ is always satisfied since the right hand side $F_e^+(\theta_2) - F_e^-(\theta_2 + \tau_e(\theta_2))$ equals $z_e(\theta_2 + \tau_e(\theta_2))$ and is therefore strictly positive by assumption. Hence, we obtain

$$T_e(\theta_1) = \theta_1 + \tau_e(\theta_1) + q_e(\theta_1) = \theta_2 + \tau_e(\theta_2) + q_e(\theta_2) = T_e(\theta_2).$$

(v) Considering two points in time $\theta_1 < \theta_2$, we show that $T_e(\theta_1) \leq T_e(\theta_2)$. Since F_e^+ is monotonically increasing, (iii) implies that

$$F_e^-(T_e(\theta_2)) = F_e^+(\theta_2) \geq F_e^+(\theta_1) = F_e^-(T_e(\theta_1)). \tag{4.10}$$

If this holds with strict inequality, we obtain by monotonicity of F_e^- that $T_e(\theta_1) < T_e(\theta_2)$. If (4.10) holds with equality we have two cases. If $z_e(\theta_2 + \tau_e(\theta_2)) > 0$, (iv) states that $T_e(\theta_1) = T_e(\theta_2)$. If $z_e(\theta_2 + \tau_e(\theta_2)) = 0$, (ii) applied to θ_1 implies that $\xi := \theta_2 + \tau_e(\theta_2) - \theta_1 - \tau_e(\theta_1) \notin [0, q_e(\theta_1))$. Since $\xi \geq 0$ by Lemma 4.1 (i) we have $\xi \geq q_e(\theta_1)$, and thus,

$$T_e(\theta_2) \overset{\text{(i)}}{=} \theta_2 + \tau_e(\theta_2) \geq \theta_1 + \tau_e(\theta_1) + q_e(\theta_1) = T_e(\theta_1).$$

(vi) The continuity of q_e follows since ν_e is always strictly positive and z_e is continuous, as it is the difference of two continuous functions. Finally, T_e is continuous since it is the sum of three continuous functions.

By (v) the function T_e is monotonically increasing for every $e \in E$, and hence, Lebesgue's theorem for the differentiability of monotone functions (Theorem 2.5 on page 19) states that T_e is almost everywhere differentiable. Since $\theta \mapsto \theta + \tau_e(\theta)$ is monotone this also holds for τ_e since it is the difference of two almost everywhere differentiable functions. As a sum of almost everywhere differentiable functions, $q_e(\theta) = T_e(\theta) - \tau_e(\theta) - \theta$ has this property as well.

(vii) The definition of $q_e(\theta)$ states that

$$\int_0^{T_e(\theta)} \nu_e(\xi)\,\mathrm{d}\xi - \int_0^{\theta+\tau_e(\theta)} \nu_e(\xi)\,\mathrm{d}\xi = z_e(\theta + \tau_e(\theta)) = F_e^+(\theta) - F_e^-(\theta + \tau_e(\theta)).$$

Taking the derivative on both sides we obtain by using the chain rule that

$$\nu_e(T_e(\theta)) \cdot T_e'(\theta) - \nu_e(\theta + \tau_e(\theta)) \cdot (1 + \tau_e'(\theta)) = f_e^+(\theta) - f_e^-(\theta + \tau_e(\theta)) \cdot (1 + \tau_e'(\theta)).$$

If $q_e(\theta) > 0$ we have by (4.3) that $f_e^-(\theta + \tau_e(\theta)) = \nu_e(\theta + \tau_e(\theta))$, and therefore, dividing by $\nu_e(T_e(\theta))$ (which is strictly positive by assumption) yields

$$T_e'(\theta) = \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))}.$$

For $q_e(\theta) = 0$ we have $f_e^-(\theta + \tau_e(\theta)) = \min\left\{ \frac{f_e^+(\theta)}{\gamma_e(\theta)}, \nu_e(\theta + \tau_e(\theta)) \right\}$ and $T_e(\theta) = \theta + \tau_e(\theta)$. Hence, dividing by $\nu_e(\theta + \tau_e(\theta)) = \nu_e(T_e(\theta))$ and using Lemma 4.1 (iii) provides

$$\begin{aligned} T_e'(\theta) &= \gamma_e(\theta) + \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} - \min\left\{ \frac{f_e^+(\theta)}{\gamma_e(\theta)}, \nu_e(T_e(\theta)) \right\} \cdot \frac{\gamma_e(\theta)}{\nu_e(T_e(\theta))} \\ &= \max\left\{ \gamma_e(\theta), \frac{f_e^+(\theta)}{\nu_e(T_e(\theta))} \right\}, \end{aligned}$$

which finishes the proof. ☐

Lemma 4.11. *An α-extension is a feasible flow over time and the extended ℓ-labels coincide with the earliest arrival times, i.e., they satisfy Equation (4.4) for all $\varphi \in [\phi, \phi + \alpha)$.*

Proof. Flow conservation on nodes, Equation (4.2), is satisfied since for all $\theta \in [\ell_v(\phi), \ell_v(\phi + \alpha))$ we have

$$\sum_{e\in\delta^+(v)} f_e^+(\theta) - \sum_{e\in\delta^-(v)} f_e^-(\theta) = \sum_{e\in\delta^+(v)} \frac{x_e'}{\ell_v'} - \sum_{e\in\delta^-(v)} \frac{x_e'}{\ell_v'} = \begin{cases} 0 & \text{if } v \in V \setminus \{s, t\} \\ r(\ell_s(\phi)) \overset{(4.8)}{=} \theta & \text{if } v = s. \end{cases}$$

Next, we show that the feasibility condition (4.3) is satisfied. For this we first consider arcs e with $x_e' > 0$, which implies $e \in E_\phi'$. By (TF3) we have that $\ell_v' \geq \gamma_e(\ell_u(\phi)) \cdot \ell_u'$. Since γ is constant during the thin flow phase, so is τ', and therefore, we have for all $\xi \in [0, \alpha)$ that

$$\begin{aligned} \ell_v(\phi + \xi) &= \ell_v(\phi) + \xi \cdot \ell_v' \\ &\geq \ell_v(\phi) + \xi \cdot \gamma_e(\ell_u(\phi)) \cdot \ell_u' \\ &\geq \ell_u(\phi) + \tau_e(\ell_u(\phi)) + \xi \cdot (1 + \tau_e'(\ell_u(\phi)) \cdot \ell_u' \\ &= \ell_u(\phi + \xi) + \tau_e(\ell_u(\phi + \xi)). \end{aligned}$$

In other words, e stays active during the thin flow phase.

We consider the outflow rate at time $\theta+\tau_e(\theta)$ for $\theta \in [\ell_u(\phi), \ell_u(\phi+\alpha))$. In the case of $\theta+\tau_e(\theta) < \ell_v(\phi)$ the feasibility condition follows from prior phases. Otherwise, $\theta + \tau_e(\theta) \in [\ell_v(\phi), \ell_v(\phi+\alpha))$, and therefore,

$$f_e^-(\theta+\tau_e(\theta)) = \frac{x'_e}{\ell'_v} \overset{\text{(TF3)}}{=} \frac{x'_e}{\rho_e(\ell'_u, x'_e)} = \begin{cases} \min\left\{ \frac{x'_e}{\gamma_e(\ell_u(\phi))\cdot \ell'_u}, \nu_e(\ell_v(\phi)) \right\} & \text{if } e \in E'_\phi \setminus E^*_\phi, \\ \nu_e(\ell_v(\phi)) & \text{else,} \end{cases}$$
$$= \begin{cases} \min\left\{ \frac{f_e^+(\theta)}{\gamma_e(\theta)}, \nu_e(\theta+\tau_e(\theta)) \right\} & \text{if } q_e(\theta) = 0, \\ \nu_e(\theta+\tau_e(\theta)) & \text{else.} \end{cases}$$

In the case that $x'_e = 0$ we either have $\ell'_v = 0$, but then there is nothing to show since the interval $[\ell_v(\phi), \ell_v(\phi+\alpha))$ would be empty, or $\ell'_v > 0$, which means by (TF2) that either e is not active, or it is active but non-resetting. In both cases we have $q_e(\ell_u(\theta)) = 0$ and since $f_e^+(\ell_u(\theta)) = 0$ for all $\theta \in [\ell_u(\phi, \ell_u(\phi+\alpha))$ the queue stays empty during this phase. (4.3) follows since $f_e^-(\theta+\tau_e(\theta)) = \frac{x'_e}{\ell'_v} = 0 = f_e^+(\theta)$ holds for all $\theta \in [\ell_u(\phi, \ell_u(\phi+\alpha))$. Altogether, we showed that the α-extension is indeed a feasible flow over time.

It remains to show that Equation (4.4) holds, which implies that the extended ℓ-functions denote the earliest arrival times. First of all we have

$$\int_0^{\ell_s(\phi+\xi)} r(\xi)\,\mathrm{d}\xi = \phi + \int_{\ell_s(\phi)}^{\ell_s(\phi+\xi)} r(\xi)\,\mathrm{d}\xi = \phi + r(\ell_s(\phi)) \cdot \ell'_s \cdot \xi = \phi + \xi.$$

Since r is always strictly positive, $\ell_s(\phi)$ is the minimal value with this property, which shows (4.4) for $v = s$. For $v \neq s$ we distinguish between three cases for every given arc $e = uv \in E$.

Case 1: $e \in E \setminus E'_\phi$.
Since α satisfies (4.6) it is satisfied for all $\xi \in [0, \alpha)$, and hence,

$$\ell_v(\phi+\xi) = \ell_v(\phi)+\xi\cdot\ell'_v \overset{(4.6)}{\leq} \ell_u(\phi)+\xi\cdot\ell'_u+\tau_e(\ell_u(\phi)+\xi\cdot\ell'_u) \leq \ell_u(\phi+\xi)+\tau_e(\ell_u(\phi+\xi)) \leq T_e(\ell_u(\phi+\xi)).$$

Case 2: $e \in E'_\phi \setminus E^*_\phi$ and $\gamma_e(\ell_u(\phi)) \cdot \ell'_u \geq \frac{x'_e}{\nu_e(\ell_v(\phi))}$.
Since e is active we have $\ell_v(\phi) = T_e(\ell_u(\phi)) = \ell_u(\phi) + \tau_e(\ell_u(\phi))$ and (TF2) implies $\ell'_v \leq \gamma_e(\ell_u(\phi)) \cdot \ell'_u$. There is no queue building up since $f_e^+(\ell_u(\phi+\xi)) = \frac{x'_e}{\ell'_u} \leq \nu_e(\ell_v(\phi))$, which means $z_e(\ell_u(\phi+\xi) + \tau_e(\ell_u(\phi))) = 0$ for all $\xi \in (0, \alpha]$. Combining these yields

$$\begin{aligned}\ell_v(\phi+\xi) \overset{\text{(TF2)}}{\leq} \ell_v(\phi) + \xi \cdot \gamma_e(\ell_u(\phi)) \cdot \ell'_u &= \ell_u(\phi) + \tau_e(\ell_u(\phi)) + \xi \cdot (1 + \tau'_e(\ell_u(\phi)) \cdot \ell'_u \\ &= \ell_u(\phi+\xi) + \tau_e(\ell_u(\phi+\xi)) \\ &= T_e(\ell_u(\phi+\xi)).\end{aligned}$$

Case 3: $e \in E^*_\phi$ or $\left(e \in E'_\phi \text{ and } \gamma_e(\ell_u(\phi)) \cdot \ell'_u < \frac{x'_e}{\nu_e(\ell_v(\phi))}\right)$.
Again, e is active, which means $\ell_v(\phi) = T_e(\ell_u(\phi))$. We have $\rho_e(\ell'_u, x'_e) = \frac{x'_e}{\nu_e(\ell_v(\phi))}$, and hence, (TF2) implies $\ell'_v \leq \frac{x'_e}{\nu_e(\ell_v(\phi))}$. Lemma 4.2 (vii) yields

$$T'_e(\ell_u(\phi)) = \frac{f_e^+(\ell_u(\phi))}{\nu_e(\ell_v(\phi))} = \frac{x'_e}{\ell'_u \cdot \nu_e(\ell_v(\phi))}$$

since either $q_e(\ell_u(\phi)) > 0$ (if $e \in E^*$) or, in the case of $e \notin E^*_\phi$, we have

$$\frac{f_e^+(\ell_u(\phi))}{\nu_e(\ell_v(\phi))} = \frac{x'_e}{\ell'_u \cdot \nu_e(\ell_v(\phi))} > \gamma_e(\ell_u(\phi)).$$

Hence, for all $\xi \in (0, \alpha]$ we obtain

$$\ell_v(\phi + \xi) \overset{\text{(TF2)}}{=} \ell_v(\phi) + \xi \cdot \ell'_v \leq \ell_v(\phi) + \xi \cdot \frac{x'_e}{\nu_e} = T_e(\ell_u(\phi)) + \xi \cdot T'_e(\ell_u(\phi)) \cdot \ell'_v = T_e(\ell_u(\phi + \xi)).$$

This shows that there is no arc with an exit time earlier than the earliest arrival time, and therefore, the left hand side of (4.4) is always smaller or equal to the right hand side.
It remains to show that the equation holds with equality. For every node $v \in V \setminus \{ s \}$ there is at least one arc $e \in E'$ with $\ell'_v = \rho_e(\ell'_u, x'_e)$ in the thin flow due to (TF2). No matter if this arc belongs to Case 2 or Case 3 the corresponding equation holds with equality, which shows for all $\xi \in (0, \alpha]$ that

$$\ell_v(\phi + \xi) = \min_{e=uv \in E} T_e(\ell_u(\phi + \xi)).$$

This shows that (4.4) is also satisfied for $v \neq s$, which completes the proof. $\square$

5 Multiple Commodities

This chapter is dedicated to networks with multiple sources and multiple sinks and the adaptation of the base model in order to handle multiple commodities. Some parts of this chapter, especially Sections 5.2 and 5.3, are based on the collaboration with Martin Skutella published in 2018 [86].

Multiple origin-destination-pairs. Nash flows over time are mainly motivated by road traffic, where, in general every road user has his or her own origin and destination. This naturally leads to the consideration of flows over time consisting of multiple commodities J, each of them with its own origin-destination-pair (s_j, t_j) and its own network inflow rate $r_j \geq 0$; see Figure 5.1. At every origin s_j, a flow enters the network with rate r_j during some interval I_j and every infinitesimally small particle of this flow has the goal to reach destination t_j as early as possible, while considering the queuing delays on the paths. Every commodity is modeled by time-dependent in- and outflow rates for every arc that must satisfy flow conservation at every node individually. Then, a dynamic equilibrium consists of a multi-commodity flow over time, where each particle chooses a convex combination of fastest routes from s_j to t_j as strategy. Cominetti, Correa and Larré proved that these dynamic equilibria exist by using infinite-dimensional variational inequalities for a path-based formulation of these flows over time [17]. Unfortunately, the techniques presented in the previous chapters for single commodity networks are not sufficient for analyzing or algorithmically constructing such multi-commodity Nash flows over time. The fact that each commodity has different earliest arrival times is the main difficulty as this causes cyclic interdependencies. Each particle entering the network has to take into account not only all flow that previously entered the network, but also flow entering the network in the future. This challenging situation is further specified in the example in Figure 5.1. Nonetheless, in Section 5.1 we will present some significant structural results for such multi-commodity Nash flows over time and we will give an arc-based proof of their

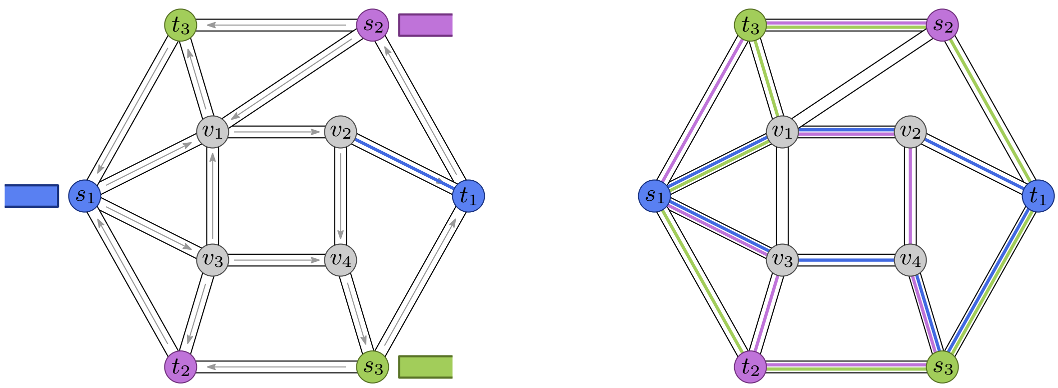


Figure 5.1: *On the left:* A multi-commodity network with three commodities. *On the right:* The particles of each commodity can choose from at least two paths. Each path overlaps with possible paths of other commodities. Hence, the waiting times a particle experiences on these links do not only depend on the flow in front of the same commodity but also on the route choices of the flows of other commodities, which may even enter the network at a later time than the particle itself. For example, the waiting time that particle $\phi = 0$ of commodity 1 experiences on arc s_3t_1, depends on the flow of commodity 3 entering at a later point in time. Their decisions, however, depend on the congestion on arc s_1v_1, which in turn might be congested by flow of commodity 1 that has entered later than particle ϕ.

existence. In order to do so we apply the techniques used in [17] to a new class of multi-commodity thin flows with resetting.

Since we were not able to show how to construct multi-commodity Nash flows over time without some strong existence theorem for infinite-dimensional variational inequalities, we consider, in addition, two special cases that can be reduced to the single-commodity setting. Networks in which all commodities share the same destination are discussed in Section 5.2 and the case that all commodities share the same origin is considered in Section 5.3. For these two special cases we essentially use some advanced super-source and super-sink constructions in order to show how to construct a Nash flow over time iteratively.

5.1 Multi-Commodity Flows Over Time

In the first part of this section we specify a proper multi-commodity flow over time model and define dynamic equilibria in this setting. Afterwards, we consider multi-commodity thin flows and prove that the derivatives of multi-commodity Nash flows over time again satisfy these flow conditions. Finally, following the lines of [17], we show the existence of dynamic equilibria in a multi-commodity setting by using an existence theorem for infinite-dimensional variational inequalities.

Note that this time we consider inflow rates with time horizons, i.e., each commodity only sends flow into the network within a finite time interval. This is a necessary condition as in the multi-commodity setting particles in the future can influence particles from the past.

5.1.1 Flow Dynamics

As before we consider a directed graph $G = (V, E)$, where each arc e is equipped with a transit time $\tau_e \geq 0$ and a capacity $\nu_e > 0$. This time, however, we also have a finite set of commodities J, each of them equipped with an origin-destination-pair $(s_j, t_j) \in V^2$ and with a network inflow rate $r_j > 0$ as well as a finite time interval I_j. We assume that there exists at least one s_j-t_j-path for every $j \in J$ and for Section 5.1 we assume that all $I_j = [a_j, b_j)$ are finite intervals.

Multi-commodity flows over time. For a multi-commodity flow over time we consider a family of locally integrable and bounded functions $f = (f^+_{j,e}, f^-_{j,e})_{j\in J, e\in E}$ where $f^+_{j,e}(\theta)$ denotes the **inflow rate** of commodity j into arc e at time θ and $f^-_{j,e}(\theta)$ denotes the respective **outflow rate**. The **cumulative in- and outflow** for each commodity j and each arc e is defined by

$$F^+_{j,e}(\theta) := \int_0^\theta f^+_{j,e}(\xi)\,\mathrm{d}\xi \quad \text{and} \quad F^-_{j,e}(\theta) := \int_0^\theta f^-_{j,e}(\xi)\,\mathrm{d}\xi,$$

and the **total (cumulative) in- and outflow rates** at each point in time θ are given by

$$f^+_e(\theta) := \sum_{j\in J} f^+_{j,e}(\theta), \quad f^-_e(\theta) := \sum_{j\in J} f^-_{j,e}(\theta), \quad F^+_e(\theta) := \sum_{j\in J} F^+_{j,e}(\theta) \quad \text{and} \quad F^-_e(\theta) := \sum_{j\in J} F^-_{j,e}(\theta).$$

Flow conservation. We say f is a **multi-commodity flow over time** if every commodity $j \in J$ conserves flow on every arc e:

$$F^-_{j,e}(\theta + \tau_e) \leq F^+_{j,e}(\theta) \qquad \text{for all } \theta \in [0, \infty), \tag{5.1}$$

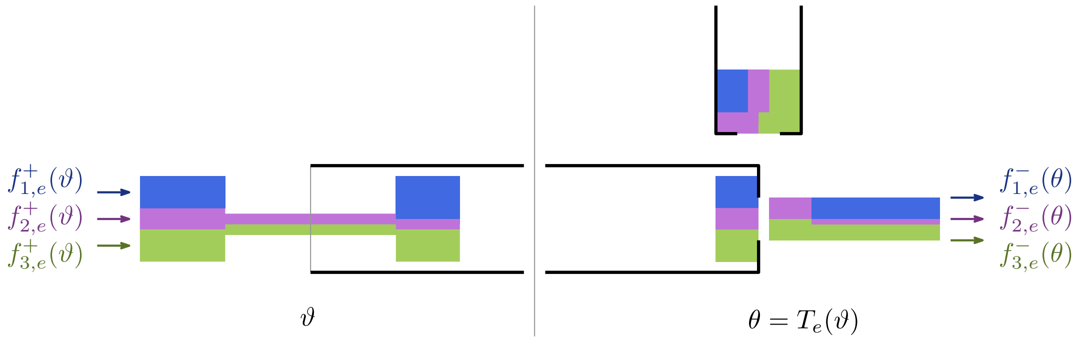


Figure 5.2: Inflow at time ϑ (*left side*) and outflow at time $\theta = T_e(\vartheta)$ (*right side*) of three commodities. We require the flow to merge perfectly, which means that the proportions of each commodity are conserved on an arc even if the flow is stretched or compressed.

and conserves flow at every node $v \in V \setminus \{ t_j \}$:

$$\sum_{e \in \delta_v^+} f^+_{j,e}(\theta) - \sum_{e \in \delta_v^-} f^-_{j,e}(\theta) = \begin{cases} 0 & \text{if } v \in V \setminus \{ s_j \} \text{ or } \theta \notin I_j \\ r_j & \text{if } v = s_j \text{ and } \theta \in I_j, \end{cases} \quad \text{for almost all } \theta \in [0, \infty). \tag{5.2}$$

Queues, waiting times and exit times. We only consider the total flow for the sizes of the queues. Hence, the queue sizes, waiting times and exit times are given by

$$z_e(\theta) := F^+_e(\theta - \tau_e) - F^-_e(\theta), \qquad q_e(\theta) := \frac{z_e(\theta + \tau_e)}{\nu_e} \quad \text{and} \quad T_e(\theta) := \theta + \tau_e + q_e(\theta).$$

Feasibility. A multi-commodity flow over time f is **feasible** if the total outflow rate follows the same flow dynamic as for a feasible single-commodity flow over time, i.e., if

$$f^-_e(\theta) = \begin{cases} \nu_e & \text{if } z_e(\theta) > 0, \\ \min \{ f^+_e(\theta - \tau_e), \nu_e \} & \text{if } z_e(\theta) = 0, \end{cases} \quad \text{for almost all } \theta \in [0, \infty). \tag{5.3}$$

Furthermore, the amount of flow of a commodity j that leaves an arc e at time θ is determined by its fraction of the total inflow rate at time ϑ when this particle entered the arc. In other words,

$$f^-_{j,e}(\theta) = \begin{cases} f^-_e(\theta) \cdot \frac{f^+_{j,e}(\vartheta)}{f^+_e(\vartheta)} & \text{if } f^+_e(\vartheta) > 0, \\ 0 & \text{else}, \end{cases} \tag{5.4}$$

where $\vartheta = \min \{ \xi \leq \theta \mid T_e(\xi) = \theta \}$ is the earliest point in time a particle can enter arc e in order to leave it at time θ. This equation ensures that arcs preserve the proportion of commodities within the flow as depicted in Figure 5.2. In particular, queues follow the first-in-first-out (FIFO) principle, which means that particles cannot overtake others within the queues.
Note that the total flow over time (f^+_e, f^-_e) is a feasible flow over time with respect to the base model, and therefore, Lemma 3.1 holds. Additionally, the follow property holds for every commodity separately.

Lemma 5.1. *For a feasible multi-commodity flow over time f we have for every arc $e \in E$, every commodity $j \in J$ and all $\theta \in [0, \infty)$ that*

$$F^+_{j,e}(\theta) = F^-_{j,e}(T_e(\theta)).$$

Proof. By Lemma 3.1 (iii) we have that $F_e^+(\xi) = F_e^-(T_e(\xi))$. Taking the derivative yields that $f_e^+(\xi) = f_e^-(T_e(\xi)) \cdot T_e'(\xi)$ for almost all $\xi \in [0, \theta]$. Hence, for $f_e^+(\xi) > 0$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi} F_{j,e}^-(T_e(\xi)) = f_{j,e}^-(T_e(\xi)) \cdot T_e'(\xi) \overset{(5.4)}{=} f_e^-(T_e(\xi)) \cdot \frac{f_{j,e}^+(\xi)}{f_e^+(\xi)} \cdot T_e'(\xi) = f_{j,e}^+(\xi). \tag{5.5}$$

In the case of $f_e^+(\xi) = 0$ both sides equal 0.
Taking the integral over $[0, \theta]$ of (5.5) yields $F_{j,e}^-(T_e(\theta)) = F_{j,e}^+(\theta)$ since $F_{j,e}^-(T_e(0)) = F_{j,e}^+(0) = 0$. $\square$

Note that (5.1) is not used in the proof. Since $F_{j,e}^+(\theta) = F_{j,e}^-(T_e(\theta))$ implies flow conservation on arcs, we can again drop Condition (5.1) for a feasible multi-commodity flow over time.

5.1.2 Multi-Commodity Nash Flows Over Time

In order to define dynamic equilibria in this setting we have to transfer the concept of current shortest paths networks and resetting arcs to the multi-commodity case.

Earliest arrival times. Since every flow commodity has its own origin we need to define earliest arrival time functions for every commodity separately. For a given flow over time f let $\ell_{j,v} \colon \mathbb{R} \to [0, \infty)$ be the earliest time a particle of commodity j can arrive at v. More precisely, we define the **earliest arrival time** for commodity $j \in J$ by

$$\begin{aligned} \ell_{j,s_j}(\phi) &\coloneqq \frac{\phi}{r_j} + a_j, \\ \ell_{j,v}(\phi) &\coloneqq \min_{e=uv \in E} T_e(\ell_{j,u}(\phi)) \qquad \text{for } v \in V \setminus \{ s_j \}. \end{aligned} \tag{5.6}$$

The flow of a commodity can be seen as an infinite long area of width 1, which means that the flow volume of an interval $[a, b] \subseteq \mathbb{R}$ equals $b - a$ (more general: the volume of a measurable subset of $\mathbb{R}$ is given by its Lebesgue-measure). Furthermore, only the particles in $K_j \coloneqq [0, (b_j - a_j) \cdot r_j)$ enters the network within the time interval I_j.
For technical reasons we also define the earliest arrival times for particles $\phi \notin K_j$ by setting $q_e(\theta) = 0$ for all $\theta < 0$. This way the earliest arrival time functions are surjective on $\mathbb{R}$.

Active and resetting arcs. We say an arc $e = uv$ is **active** for particle ϕ and commodity j if

$$\ell_{j,v}(\phi) = T_e(\ell_{j,u}(\phi))$$

and we denote the set of all active arcs for ϕ and j by

$$E'_{j,\phi} \coloneqq \{ e = uv \in E \mid \ell_{j,v}(\phi) = \ell_{j,u}(\phi) + \tau_e + q_e(\ell_{j,u}(\phi)) \}.$$

The graph $G_{j,\phi} \coloneqq (V, E'_{j,\phi})$ is called the **current shortest paths network** of particle ϕ and commodity j. Furthermore, we define

$$E^*_{j,\phi} \coloneqq \{ e = uv \in E \mid q_e(\ell_{j,u}(\phi)) > 0 \}$$

to be the **resetting** arcs for particle ϕ and commodity j.
Note that in this setting there might be arcs that are resetting but not active for a particle of some commodity.

Dynamic equilibria. We can now transfer the definition of Nash flows over time to this multi-commodity setting.

Definition 5.2 (Multi-commodity Nash flow over time). A feasible multi-commodity flow over time f is a **multi-commodity Nash flow over time** if

$$f^+_{j,e}(\theta) > 0 \quad \Rightarrow \quad \theta \in \ell_{j,u}(\Phi_{j,e}) \qquad \text{for all } e = uv \in E, j \in J \text{ and almost all } \theta \in [0,\infty), \quad \text{(mcN)}$$

where $\Phi_{j,e} := \{\, \phi \in \mathbb{R}_{\geq 0} \mid e \in E'_{j,\phi} \,\}$ is the set of flow particles of commodity j for which arc e is active.

We can characterize Nash flows over time in the multi-commodity setting as follows.

Lemma 5.3. *For a feasible multi-commodity flow over time f the following statements are equivalent:*

(i) f is a multi-commodity Nash flow over time.

(ii) $F^+_{j,e}(\ell_{j,u}(\phi)) = F^-_{j,e}(\ell_{j,v}(\phi))$ for all $e = uv$, $j \in J$ and all $\phi \in \mathbb{R}_{\geq 0}$.

The proof of this lemma is similar to the proof of the corresponding part in Lemma 3.3. It can be found in the appendix on page 89.
In a multi-commodity Nash flow over time the waiting times, and therefore the active and resetting arcs, are completely characterized by the earliest arrival time functions.

Lemma 5.4. *Given a multi-commodity Nash flow over time f with arrival time functions $(\ell_{j,v})_{j\in J, v\in V}$, we have for all arcs $e = uv \in E$ and all $\theta \in [0,\infty)$ that*

$$q_e(\theta) = \max_{j\in J} \left(\max\{\, \ell_{j,v}(\phi_j) - \ell_{j,u}(\phi_j) - \tau_e, 0 \,\}\right) \qquad \textit{with} \qquad \phi_j := \min\{\, \phi \in \mathbb{R}_{\geq 0} \mid \ell_{j,u}(\phi) = \theta \,\}.$$

Proof. If $q_e(\theta) = 0$ we have by (5.6) for all commodities j that

$$\ell_{j,v}(\phi_j) \leq \ell_{j,u}(\phi_j) + \tau_e + q_e(\ell_{j,u}(\phi_j)) = \ell_{j,u}(\phi_j) + \tau_e.$$

For $q_e(\theta) > 0$ we show that there has to be at least one commodity $j \in J$ for which e is active for particle ϕ_j. Let j be the commodity for which e was active at the latest point in time before θ, i.e.,

$$j := \arg\max_{i\in J} \ell_{i,u}(\varphi_i) \qquad \text{with} \qquad \varphi_i := \max\{\, \xi \leq \phi_j \mid e \in E'_{i,\xi} \,\}.$$

Since no flow was sent into e between $\ell_{j,u}(\varphi_j)$ and $\theta = \ell_{j,u}(\phi_{j,u})$ we obtain for the total cumulative inflow that $F^+_e(\ell_{j,u}(\phi_j)) - F^+_e(\ell_{j,u}(\varphi_j)) = 0$. Hence, by Lemma 3.1 (iv) we get

$$\ell_{j,v}(\phi_j) \leq T_e(\ell_{j,u}(\phi_j)) = T_e(\ell_{j,u}(\varphi_j)) = \ell_{j,v}(\varphi_j) \leq \ell_{j,v}(\phi_j).$$

Thus, we have equality, which shows that e is active for ϕ_j. It follows that

$$q_e(\theta) = q_e(\ell_{j,u}(\phi_j)) = \ell_{j,v}(\phi_j) - \ell_{j,u}(\phi_j) - \tau_e.$$

Clearly, there cannot be another commodity j' with $\ell_{j',v}(\phi_{j'}) - \ell_{j',u}(\phi_{j'}) - \tau_e > q_e(\theta)$ since this would contradict the definition of the earliest arrival times in Equation (5.6). □

Underlying static flows. We define the **underlying static flow** for each commodity j by

$$x_{j,e}(\phi) := F^+_{j,e}(\ell_{j,u}(\phi)) = F^-_{i,e}(\ell_{i,v}(\phi)) \qquad \text{for all } e = uv \in E.$$

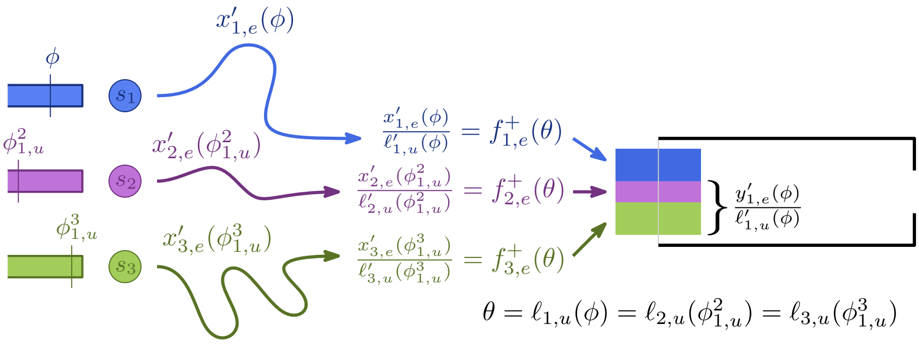


Figure 5.3: Foreign flow entering an arc. Particle ϕ of commodity 1 is entering arc $e = uv$ at time $\theta := \ell_{1,u}(\phi)$. To determine the inflow rate of the other commodities at this point in time, we consider the particles $\phi^i_{1,u}$ that also reach node u at time θ. The value $x'_{i,e}(\phi^i_{1,u})$ denotes the part of the flow of commodity i that will use arc e. Hence, we obtain the foreign inflow rates at time θ by dividing this value by $\ell'_{i,u}(\phi^i_{1,u})$.

It is easy to see that the arc vector $(x_j(\phi))_{e\in E}$ forms a static s_j-t_j-flow of value ϕ if $\phi \in K_j$. Furthermore, these functions are monotone and almost everywhere differentiable and the vector of derivatives $(x'_j(\phi))_{e\in E}$ forms a static s_j-t_j-flow of value 1 for $\phi \in K_j$ or of value 0 otherwise.

Underlying foreign flows. The main challenge of multi-commodity dynamic equilibria is that the stress of an arc, and therefore the route choice of each particle, depends on flow of all commodities simultaneously. To obtain some structural insight nevertheless we define the **underlying foreign flow** by

$$y_{j,e}(\phi) := \sum_{i\in J\setminus\{\,j\,\}} F^+_{i,e}(\ell_{j,u}(\phi)) \qquad \text{for all } e = uv \in E.$$

Note that this is not a static flow in general since the cumulative inflow $F^+_{i,e}(\ell_{j,u}(\phi))$ of some commodity i into an arc e that is active for commodity i but not for commodity j generally differs from the cumulative outflow $F^-_{i,e}(\ell_{j,v}(\phi))$.
Nonetheless, we have

$$y_{j,e}(\phi) = \sum_{i\in J\setminus\{\,j\,\}} x_{i,e}(\phi^i_{j,u}) \qquad \text{with} \qquad \phi^i_{j,u} := \min \ell^{-1}_{i,u}(\ell_{j,u}(\phi)).$$

Note that $\phi^i_{j,u}$ is the very first particle of commodity i that can arrive at u (when taking a shortest path) exactly at the time when the particle ϕ of commodity j reaches u. It is, therefore, a function in dependency of ϕ, but for sake of readability we omit the parameter in most cases.

Lemma 5.5. *For all $j \in J$ and $e \in E$ the underlying foreign flow $y_{j,e}(\phi)$ is almost everywhere differentiable with*

$$y'_{j,e}(\phi) = \sum_{i\in J\setminus\{\,j\,\}} f^+_{i,u}(\ell_{j,u}(\phi)) \cdot \ell'_{j,u}(\phi) = \begin{cases} \sum_{i\in J\setminus\{\,j\,\}} x'_{i,e}(\phi^i_{j,u}) \cdot \frac{\ell'_{j,u}(\phi)}{\ell'_{i,u}(\phi^i_{j,u})} & \text{if } \ell'_{j,u}(\phi) > 0, \\ 0 & \text{else.} \end{cases}$$

An illustration of the relation between the foreign inflow rates and the derivatives of the underlying foreign flow can be found in Figure 5.3.

Proof of Lemma 5.5. By Lebesgue's theorem for the differentiability of monotone functions, $\phi^i_{j,u}(\phi)$ is almost everywhere differentiable as it is monotone. As a composition and sum of almost everywhere differentiable functions so is $y_{j,e}(\phi)$. Let ϕ be a particle such that the functions $\ell_{j,u}(\phi)$, $\phi^i_{j.u}(\phi)$,

$\ell_{i,u}(\phi^i_{j,u}(\phi))$ and $y_{j,e}(\phi)$ are differentiable for all $i \in J \setminus \{ j \}$. This is given for almost all particles. The first equation follows immediately by the chain rule. For $\ell'_{j,u}(\phi) = 0$ we have $y'_{j,e}(\phi) = 0$. So let us suppose that $\ell'_{j,u}(\phi) > 0$. We obtain

$$0 < \ell'_{j,u}(\phi) = \frac{\mathrm{d}}{\mathrm{d}\phi}\ell_{i,u}(\phi^i_{j,u}(\phi)) = \ell'_{i,u}(\phi^i_{j,u}(\phi)) \cdot \frac{\mathrm{d}}{\mathrm{d}\phi}\phi^i_{j,u}(\phi),$$

and therefore, $\ell'_{i,u}(\phi^i_{j,u}(\phi)) > 0$. Again with the chain rule and the equation above it follows immediately that

$$y'_{j,e}(\phi) = \sum_{i \in J\setminus\{ j \}} x'_{i,e}(\phi^i_{j,u}(\phi)) \cdot \frac{\mathrm{d}}{\mathrm{d}\phi}\phi^i_{j,u}(\phi) = \sum_{i \in J\setminus\{ j \}} x'_{i,e}(\phi^i_{j,u}(\phi)) \cdot \frac{\ell'_{j,u}(\phi)}{\ell'_{i,u}(\phi^i_{j,u}(\phi))}.$$

□

5.1.3 Multi-Commodity Thin Flows

We now want to describe the structure of the derivatives of the underlying static flow (the strategy of the particles) by an extended thin flow formulation. However, we have to include the derivatives of the foreign flow into our consideration, and since the foreign flow heavily depends on the underlying static flow of other commodities we cannot consider only one particle (or one interval of particles) at a time, but we have to consider the strategy of all particles simultaneously.

Definition 5.6 (Multi-commodity thin flow).

For a given family of arc functions $x' = (x'_{j,e})_{j \in J, e \in E}$ and node functions $\ell' = (\ell'_{j,v})_{j \in J, v \in V}$ we define $E'_{j,\phi}, E^*_{j,\phi} \subseteq E$ and $y'_{j,e} \colon \mathbb{R}_{\geq 0} \to [0, \infty)$ for all $j \in J$, $\phi \in \mathbb{R}_{\geq 0}$ and $e \in E$ as described above in dependency of the functions $\ell_{j,v}(\phi) := \int_0^\phi \ell'_{j,v}(\xi)\,\mathrm{d}\xi$. We say that the pair (x', ℓ') forms a **multi-commodity thin flow** if the following conditions are satisfied:

For all $\phi \in \mathbb{R}_{\geq 0}$ the arc vector $(x'_{j,e}(\phi))_{e \in E}$ forms a static s_j-t_j-flow of value 1 if $\phi \in K_j$ or of value 0 if $\phi \notin K_j$. In both cases we have $x'_{j,e}(\phi) = 0$ for all $e \notin E'_{j,\phi}$ and for almost all $\phi \in \mathbb{R}_{\geq 0}$ the following equations hold:

$$\ell'_{j,s_j}(\phi) = \frac{1}{r_j} \qquad \text{for all } j \in J, \qquad \text{(mcTF1)}$$

$$\ell'_{j,v}(\phi) = \min_{e = uv \in E'_{j,\phi}} \rho^\phi_{j,e}\left(\ell'_{j,u}(\phi), x'_{j,e}(\phi), y'_{j,e}(\phi)\right) \qquad \text{for all } j \in J, v \in V \setminus \{ s_j \}, \qquad \text{(mcTF2)}$$

$$\ell'_{j,v}(\phi) = \rho^\phi_{j,e}\left(\ell'_{j,u}(\phi), x'_{j,e}(\phi), y'_{j,e}(\phi)\right) \qquad \text{for all } j \in J, e = uv \in E'_{j,\phi} \text{ with } x'_{j,e} > 0, \qquad \text{(mcTF3)}$$

$$\text{where} \qquad \rho^\phi_{j,e}(\ell'_{j,u}, x'_{j,e}, y'_{j,e}) := \begin{cases} \frac{x'_{j,e} + y'_{j,e}}{\nu_e} & \text{if } e = uv \in E^*_{j,\phi}, \\ \max\left\{ \ell'_{j,u}, \frac{x'_{j,e} + y'_{j,e}}{\nu_e} \right\} & \text{if } e = uv \in E'_{j,\phi} \backslash E^*_{j,\phi}. \end{cases}$$

The first main result for multi-commodity Nash flows over time states that the derivatives form a multi-commodity thin flow.

Theorem 5.7.

For a multi-commodity Nash flow over time f, the derivatives $(x'_{j,e})_{j \in J, e \in E}$ and $(\ell'_{j,v})_{j \in J, v \in V}$ form a multi-commodity thin flow.

The proof follows the line of the proof of Theorem 3.6 incorporating the derivatives of the foreign flow. It can be found in the appendix on page 90.
For the reverse direction we show that for a given multi-commodity thin flow (x', ℓ') we can reconstruct the Nash flow over time by setting

$$f^+_{j,e}(\theta) := \frac{x'_{j,e}(\phi)}{\ell'_{j,u}(\phi)} \quad \text{for } \theta = \ell_{j,u}(\phi) \qquad \text{and} \qquad f^-_{j,e}(\theta) := \frac{x'_{j,e}(\phi)}{\ell'_{j,v}(\phi)} \quad \text{for } \theta = \ell_{j,v}(\phi)$$

for all $\phi \in \mathbb{R}_{\geq 0}$ and every $e = uv \in E$. Furthermore, we set $f^+_{j,e}(\theta) = 0$ for $\theta < \ell_{j,u}(0)$ and $f^-_{j,e}(\theta) = 0$ for $\theta < \ell_{j,v}(0)$.

Theorem 5.8.

For every multi-commodity thin flow (x', ℓ') the family of functions $f = (f^+_{j,e}, f^-_{j,e})_{j \in J, e \in E}$ as defined above is a multi-commodity Nash flow over time with earliest arrival time functions

$$\ell_{j,v}(\phi) := \int_0^\phi \ell'_{j,v}(\xi)\,\mathrm{d}\xi \qquad \textit{for all } j \in J, v \in V \textit{ and } \phi \in \mathbb{R}_{\geq 0}.$$

Proof. Clearly, (5.2) is satisfied since flow of every commodity j is conserved at every node v at every point in time $\theta = \ell_{j,v}(\phi)$, i.e.,

$$\sum_{e \in \delta^+_v} f^+_{j,e}(\theta) - \sum_{e \in \delta^-_v} f^-_{j,e}(\theta) = \sum_{e \in \delta^+_v} \frac{x'_{j,e}(\phi)}{\ell'_{j,v}(\phi)} - \sum_{e \in \delta^-_v} \frac{x'_{j,e}(\phi)}{\ell'_{j,v}(\phi)} = \begin{cases} 0 & \text{if } v \in V \setminus \{\, s_j \,\} \text{ or } \phi \notin K_j, \\ r_j & \text{if } v = s_j \text{ and } \phi \in K_j. \end{cases}$$

Note that for $v = s_j$ we have $\phi \in K_j$ if, and only if, $\theta = \ell_{j,s_j}(\phi) \in I_j$.
For a given $e = uv \in E$ and $\theta \in [0, \infty)$ let $\phi_j \in \mathbb{R}_{\geq 0}$ such that $\ell_{j,u}(\phi_j) = \theta$ for all $j \in J$. Considering the commodities j, where e is active for j and ϕ_j, we observe that also all $\ell_{j,v}(\phi_j)$ of these commodities coincide. Hence, (mcTF3) yields

$$\begin{aligned} f^-_e(\theta) = \sum_{j \in J} \frac{x'_{j,e}(\phi_j)}{\ell'_{j,v}(\phi_j)} &= \begin{cases} \nu_e & \text{if } e \in E^*_{j,\phi_j} \text{ for some } j \text{ with } e \in E'_{j,\phi_j}, \\ \min\left\{ \sum_{j \in J} \frac{x'_{j,e}(\phi_j)}{\ell'_{j,u}(\phi_j)}, \nu_e \right\} & \text{else,} \end{cases} \\ &= \begin{cases} \nu_e & \text{if } q_e(\theta) > 0, \\ \min\{\, f^+_e(\theta), \nu_e \,\} & \text{else.} \end{cases} \end{aligned}$$

This shows (5.3).
Equation (5.4) follows by Lemma 5.5 since

$$f^-_{j,e}(\theta) = \frac{x'_{j,e}(\phi_j)}{\ell'_{j,v}(\phi_j)} = \frac{y'_{j,e}(\phi_j) + x'_{j,e}(\phi_j)}{\ell'_{j,v}(\phi_j)} \cdot \frac{x'_{j,e}(\phi_j)}{\ell'_{j,u}(\phi_j)} \cdot \frac{\ell'_{j,u}(\phi_j)}{y'_{j,e}(\phi_j) + x'_{j,e}(\phi_j)} = f^-_e(\theta) \cdot \frac{f^+_{i,e}(\theta)}{f^+_e(\theta)}.$$

In order to show that the ℓ-functions satisfy Equation (5.6) we prove that the derivatives of $\ell_{j,v}(\phi)$ and of $\min_{e = uv \in E} T_e(\ell_{j,u}(\phi))$ coincide for all $\phi \in \mathbb{R}_{\geq 0}$. Lemma 3.1 (vii) implies for almost all $\theta \in [0, \infty)$ that

$$T'_e(\theta) = 1 + q'_e(\theta) = \begin{cases} \max\left\{ \frac{f^+_e(\theta)}{\nu_e}, 1 \right\} & \text{if } q_e(\theta) = 0, \\ \frac{f^+_e(\theta)}{\nu_e} & \text{else.} \end{cases}$$

Hence,

$$T'_e(\ell_{j,u}(\phi)) \cdot \ell'_{j,u}(\phi) = \begin{cases} \max\left\{ \frac{x'_{j,e}(\phi)+y'_{j,e}(\phi)}{\nu_e}, \ell'_{j,u}(\phi) \right\} & \text{if } q_e(\ell_{j,u}(\phi)) = 0, \\ \frac{x'_{j,e}(\phi)+y'_{j,e}(\phi)}{\nu_e} & \text{else.} \end{cases}$$

This together with (mcTF2) and the differentiation rule for a minimum (Lemma 2.3 on page 17) implies that

$$\ell'_{j,v}(\phi) = \frac{\mathrm{d}}{\mathrm{d}\phi} \min_{e=uv\in E} T_e(\ell_{j,u}(\phi)).$$

By Lebesgue's differentiation theorem (Theorem 2.4 on page 18) we obtain (5.6). In other words, the ℓ-functions are indeed the earliest arrival times for the constructed feasible flow over time f. Finally, f is a multi-commodity Nash flow over time by Lemma 5.3 since

$$F^+_{j,e}(\ell_{j,u}(\phi)) = \int_0^\phi f^+_{j,e}(\ell_{j,u}(\xi))\cdot\ell'_{j,u}(\xi)\,\mathrm{d}\xi = \int_0^\phi x'_{j,e}(\xi)\,\mathrm{d}\xi = \int_0^\phi f^-_{j,e}(\ell_{j,v}(\xi))\cdot\ell'_{j,v}(\xi)\,\mathrm{d}\xi = F^-_{j,e}(\ell_{j,v}(\phi))$$

for all $e = uv \in E$, $j \in J$ and $\phi \in \mathbb{R}_{\geq 0}$. □

To sum this up, Theorems 5.7 and 5.8 show that multi-commodity Nash flows over time correspond one-to-one to multi-commodity thin flows.

5.1.4 Existence of Multi-Commodity Nash Flows Over Time

The existence of dynamic equilibria in a multi-commodity setting was first shown by Cominetti, Correa and Larré in [17], even though the proof is not worked out in this paper. The main idea is to represent feasible multi-commodity flows over time in a path-based formulation, i.e., as vectors of inflow functions in the L^p-space, and then to formulate the Nash flow condition as an infinite-dimensional variational inequality. Using Brézis' theorem (Theorem 2.8 on page 21) guarantees the existence of a multi-commodity Nash flow over time. We will present a complete proof for this, but instead of representing the flow over time path-based we will show the existence of a multi-commodity thin flow in its arc-based form.
In order to avoid degenerated cases we assume from now on that all transit times are strictly positive. Let $H > 0$ such that $K_j \subseteq [0, H]$ for all $j \in J$. For this we represent the flow by a vector of functions $x' = (x'_{j,e})_{j\in J, e\in E} \in L^2([0, H])^{J\times E}$. Recall that this is a Hilbert space with scalar product

$$\langle x, y\rangle := \sum_{j\in J, e\in E} \int_0^H x_{j,e}(\xi)\cdot y_{j,e}(\xi)\,\mathrm{d}\xi.$$

Variational inequalities. As described in Section 2.5.2 on page 20 we want to utilize the following infinite-dimensional variational inequality. For a set of function vectors $X \subseteq L^2([0, H])^{J\times E}$ and a mapping $\mathcal{A}\colon X \to L^2([0, H])^{J\times E}$ the variational inequality is the following task.

$$\text{Find } x' \in X \text{ such that} \quad \langle \mathcal{A}(x'), z' - x'\rangle \geq 0 \quad \text{for all } z' \in X. \tag{infVI}$$

As long as we define X to be non-empty, closed, convex and bounded and the mapping $\mathcal{A}$ to be weak-strong continuous, Brézis' theorem (Theorem 2.8 on page 21) guarantees a solution to this variational inequality.

We start by defining

$$X := \left\{ (x'_{j,e})_{j\in J, e\in E} \in L^2([0,H])^{J\times E} \;\middle|\; \begin{array}{c} (x'_{j,e}(\phi))_{e\in E} \text{ is a static } s_j\text{-}t_j\text{-flow of} \\ \text{value 1 for } \phi \in K_j \text{ and 0 for } \phi \notin K_j. \end{array} \right\}.$$

Clearly, X is a non-empty, closed, convex and bounded subset of $L^2([0,H])^{J\times E}$ since a convex combination of two static flows with the same value is again a static flow of this value.
In order to define the mapping $\mathcal{A}$ we first need the following lemma.

Lemma 5.9. *For every $x' = (x'_{j,e})_{j\in J, e\in E} \in X$ we can construct a vector $(\ell_{j,v})_{j\in J, v\in V}$ of continuous and monotonically increasing functions such that their derivatives $(\ell'_{j,v})_{j\in J, v\in V}$ satisfy* (mcTF1) *and* (mcTF2) *for all $\phi \in [0,H]$, where we define*

$$\phi^i_{j,u}(\phi) := \min\{\, \varphi \geq 0 \mid \ell_{i,u}(\varphi) = \ell_{j,u}(\phi) \,\} \quad \textit{and} \quad y'_{j,e}(\phi) := \sum_{i\in J\setminus\{j\}} x'_{i,e}(\phi^i_{j,u}(\phi)) \cdot \frac{\ell'_{j,u}(\phi)}{\ell'_{i,u}(\phi^i_{j,u}(\phi))}.$$

Furthermore, the mapping $(x'_{j,e})_{j\in J, e\in E} \mapsto (\ell_{j,u})_{j\in J, u\in V}$ is weak-strong continuous.

The key idea of the proof is to start at time 0 and then extend these functions step by step for later points in time. Note that we extend over the range of the functions and not over the domain. In each extension step we determine the change of $\ell_{j,v}$ by plugging $x'_{j,e}(\ell_u(\phi))$ and $y'_{j,e}(\ell_u(\phi))$ into (mcTF2). As we assumed that the transit time of every arc is positive we have that $\ell_{j,u}(\phi) < \ell_{j,v}(\phi) = \theta$ for all arcs that are active for j and ϕ. Hence, we can extend these functions at least by the minimal transit time in every step. A detailed proof can be found in the appendix on page 91.
Lemma 5.9 shows that for a given $x' \in X$ we obtain functions $\ell_{j,v}$, which we can plug into the equation of Lemma 5.4 in order to obtain waiting time functions $(q_e)_{e\in E}$. Note that the mapping $x' \mapsto q$ is also weak-strong continuous. Furthermore, the ℓ-functions satisfy Equation (5.6), as we have already shown in the second half of the proof of Theorem 5.8 (we do not use (mcTF3) in this part of the proof). It is worth noting, however, that these ℓ- and q-functions do not belong to a feasible flow over time, in general, as flow conservation might not hold when deriving in- and outflow rate functions in the usual way.
Finally, we can define the weak-strong continuous mapping $\mathcal{A}\colon X \to L^2([0,H])^{J\times E}$ by

$$(x'_{j,e})_{j\in J, e\in E} \mapsto (h_{j,e})_{j\in J, e\in E} \quad \text{with} \quad h_{j,e}(\phi) := \ell_{j,u}(\phi) + \tau_e + q_e(\ell_{j,u}(\phi)) - \ell_{j,v}(\phi).$$

In other words, if x' corresponds to a feasible flow over time, $h_{j,e}(\phi)$ denotes the delay of particle ϕ when traveling as fast as possible to u first and then using arc e, instead of taking the fastest direct route to v. In a Nash flow over time this value should always be 0 for each arc e with $x'_{j,e}(\phi) > 0$.

Theorem 5.10.

For every multi-commodity network with positive transit times there exists a multi-commodity thin flow, and hence, a multi-commodity Nash flow over time.

Proof. Let x' be a solution to the variational inequality constructed above, which exists due to Theorem 2.8. In other words, it holds that

$$\sum_{e\in E, j\in J} \int_0^H (\ell_{j,u}(\phi) + \tau_e + q_e(\ell_{j,u}(\phi)) - \ell_{j,v}(\phi)) \cdot (z'_{j,e} - x'_{j,e})\, \mathrm{d}\xi \geq 0 \qquad \text{for } z' \in X.$$

Let $(\ell_v)_{v \in V}$ be the node labels corresponding to x' according to Lemma 5.9 with derivatives $(\ell'_v)_{v \in V}$. We will show that (x', ℓ') satisfies the multi-commodity thin flow conditions for $\phi \in [0, H]$.

As (mcTF1) and (mcTF2) hold for $(\ell'_v)_{v \in V}$ by Lemma 5.9 it only remains to show that (mcTF3) holds for almost all $\phi \in [0, H]$. In order to do so, suppose that there exist a commodity j, an arc $e = uv$ and a set with positive measure $\Phi \subseteq [0, H]$ such that $x'_{j,e}(\phi) > 0$ and

$$\ell'_{j,v}(\phi) < \rho^{\phi}_{j,e}\left(\ell'_{j,u}(\phi), x'_{j,e}(\phi), y'_{j,e}(\phi)\right).$$

We assume that Φ is contained in a small interval $[a, b]$ and that $x'_{j,e}(\phi) \geq \varepsilon$ for some $\varepsilon > 0$.

Note that for every $\phi \in \Phi$ there are two s_j-t_j-paths P_ϕ, Q_ϕ, which satisfy the following conditions. Firstly, we require that $e \in P_\phi$ and $x'_{j,e'}(\phi) > \varepsilon$ for all $e' \in P_\phi$, and secondly, for all $e' = u'v' \in Q_\phi$ we demand that $e' \in E'_{j,\phi}$ as well as

$$\ell'_{j,v'}(\phi) = \rho^{\phi}_{j,e'}\left(\ell'_{j,u'}(\phi), x'_{j,e'}(\phi), y'_{j,e'}(\phi)\right).$$

The existence of P_ϕ follows by the flow conservation of the static flow $x'_{j,e}(\phi)$ (ε can be redefined to be small enough) and the existence of Q_ϕ follows by the construction of the ℓ'-functions.

It is possible to partition Φ into measurable sets such that the particles ϕ of each subset have the same paths-pair (P_ϕ, Q_ϕ). Thus, at least one of these subsets has to have a positive measure, and hence, without loss of generality, we can assume that all particles in Φ have the same pair of paths, which we denote by P and Q.

We set $z' := x'$ with the exception of $z'_{j,e'}(\phi) := x'_{j,e'}(\phi) - \varepsilon$ for all $e' \in P$. Furthermore, let $\phi \in \Phi$ and $z'_{j,e'}(\phi) := x'_{j,e'}(\phi) + \varepsilon$ for all $e' \in Q$ and $\phi \in \Phi$. Clearly, $z' \in X$, as the small shift of flow from P to Q, does not violate the flow conservation and does not change the total flow value. We obtain that

$$\begin{aligned}
\langle \mathcal{A}(x'), z' - x' \rangle &= \sum_{e \in E, j \in J} \int_0^H T_e(\ell_{j,u}(\phi)) \cdot (z'_{j,e} - x'_{j,e}) \,\mathrm{d}\xi \\
&= -\varepsilon \cdot \sum_{e' = u'v' \in P} \int_\Phi T_{e'}(\ell_{j,u'}(\phi)) - \ell_{j,v'}(\phi) \,\mathrm{d}\phi + \varepsilon \cdot \sum_{e' \in Q} \int_\Phi T_{e'}(\ell_{j,u'}(\phi)) - \ell_{j,v'}(\phi) \,\mathrm{d}\phi \\
&\leq -\varepsilon \cdot \int_\Phi T_e(\ell_{j,u}(\phi)) - \ell_{j,v}(\phi) \,\mathrm{d}\phi + \varepsilon \cdot \sum_{e' \in Q} \int_\Phi T_{e'}(\ell_{j,u'}(\phi)) - \ell_{j,v'}(\phi) \,\mathrm{d}\phi \\
&= -\varepsilon \cdot \int_\Phi T_e(\ell_{j,u}(\phi)) - \ell_{j,v}(\phi) < 0.
\end{aligned}$$

The first inequality follows, since ℓ satisfies (5.6), and hence, $T_{e'}(\ell_{j,u'}(\phi)) - \ell_{j,v'} \geq 0$ for all $e' = u'v' \in E$. The last equation holds, since Q is a path of active arcs for all particles in Φ, and therefore, $T_{e'}(\ell_{j,u'}(\phi)) - \ell_{j,v'}(\phi) = 0$ for all $e' = u'v' \in Q$ and all $\phi \in \Phi$. But this is a contradiction to the variational inequality (infVI). Hence, (x', ℓ') satisfies the thin flow conditions for almost all $\phi \in [0, H]$.

This shows the existence of a multi-commodity thin flow and with Theorem 5.8 it follows that there also exists a multi-commodity Nash flow over time on every multi-commodity network. □

5.2 Common Destination

Even though multi-commodity Nash flows over time exist, we do not know how to construct them, as exact solutions to infinite-dimensional variational inequalities cannot be computed algorithmically. In order to transfer the concepts of the base model to a multi-terminal setting, we consider the

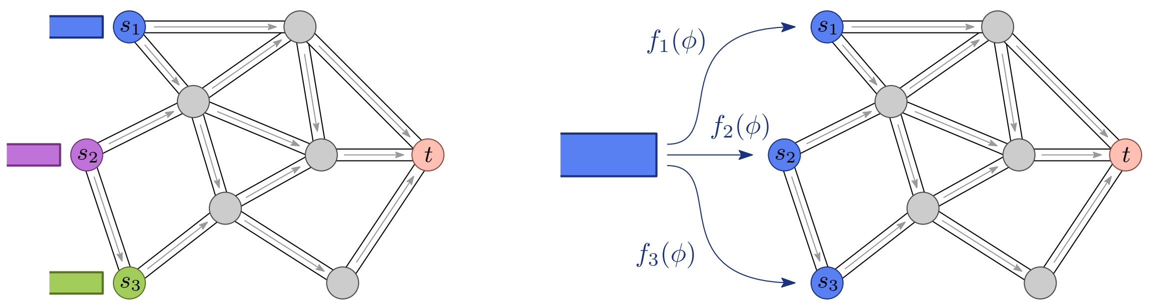


Figure 5.4: *On the left:* A multi-commodity network where each commodity has the same destination t. *On the right:* In this case we can construct Nash flows over time by reducing the problem to a single commodity problem. The inflow distribution denotes the proportion of the flow that enters through each source.

special case where all commodities have the same destination. In other words, we have multiple sources but only one sink. We consider only one commodity, i.e., only one flow, where each particle can choose the source for entering the network; see Figure 5.4. Sources further away from the sink might not be chosen by particles with high priority, but their network inflow rates will be maximal from time 0 onwards nonetheless.

Evacuation scenarios. Even though this setting seems to be artificial at first glance it has an important and relevant application. Imagine an inhabited region with a high risk of flooding, where in the case of rising water levels everyone tries to reach a high-altitude shelter as fast as possible. Since the road users seek for protection and do not care at which of the shelters they end up, we can connect all nodes representing one of these safe places to a super sink. Without regulations an evacuation now corresponds to a Nash flow over time with multiple sources but only a single sink, as everyone starts at their home and tries to reach one of the shelters as fast as possible.

Networks. To model this special case we consider a network $G = (V, E)$ with transit times $\tau_e \geq 0$, capacities $\nu_e > 0$ and a sink node t as before. But this time we have, in addition, a set of sources $S := \{ s_j \mid j \in J \}$ with network inflow rates r_j for each commodity j. We assume that every source can reach the sink, that every node is reachable by at least one source and that all directed cycles have a strictly positive total transit time.

We do not distinguish between different commodities because as soon as the particles have entered the network they all have the same goal, namely to reach the sink as fast as possible, and therefore, their identity is interchangeable.

We use the same notation of flow rates, cumulative flows, queue sizes and waiting times as in the base model, this time, however, we say that flow is conserved on a node $v \in V \setminus \{ t \}$ if

$$\sum_{e \in \delta_v^+} f_e^+(\theta) - \sum_{e \in \delta_v^-} f_e^-(\theta) = \begin{cases} 0 & \text{if } v \in V \setminus S, \\ r_j & \text{if } v = s_j \in S. \end{cases} \tag{5.7}$$

A **flow over time** is a family of locally integrable and bounded functions $(f_e^+, f_e^-)_{e \in E}$ that satisfies (3.1) and (5.7) and it is **feasible** if (3.3) is fulfilled.

Inflow distributions. In order to denote which particle enters through which source we need additional functions. A family of locally integrable functions $f_j \colon \mathbb{R}_{\geq 0} \to [0, 1]$, for $j \in J$, is called **inflow distribution** if $\sum_{j \in J} f_j(\phi) = 1$ for almost all $\phi \in \mathbb{R}_{\geq 0}$ and if each **cumulative source inflow** $F_j(\phi) := \int_0^\phi f_j(\varphi) \, \mathrm{d}\varphi$ is unbounded for $\phi \to \infty$. The function $f_j(\phi)$ describes the fraction of particle ϕ

that enters the network trough s_j. The cumulative source inflow functions have to be unbounded in order to guarantee that the inflow rates at the sources never run dry.

Source arrival times. Given a feasible flow over time f, the **source arrival time** functions map each particle $\phi \in \mathbb{R}_{\geq 0}$ to the time it arrives at s_j and they are given by

$$T_j(\phi) := \frac{F_j(\phi)}{r_j}.$$

Earliest arrival times. The **earliest arrival times** are now defined by

$$\begin{aligned} \ell_{s_j}(\phi) &= \min\left(\{\, T_j(\phi)\,\} \cup \{\, T_e(\ell_u(\phi)) \mid e = us_j \in E\,\}\right) && \text{for } j \in J, \\ \ell_v(\phi) &= \min_{e=uv\in E} T_e(\ell_u(\phi)) && \text{for } v \in V\backslash S. \end{aligned} \tag{5.8}$$

This is well-defined since all cycles in G have positive travel times by assumption.

Current shortest paths networks and active arcs. As before we call an arc $e = uv$ **active** for ϕ if $\ell_v(\phi) = T_e(\ell_u(\phi))$ holds and we denote the set of all active arcs for a particle ϕ by E'_ϕ as well as the **current shortest paths network** by $G'_\phi = (V, E'_\phi)$. Furthermore, $E^*_\phi := \{e = uv \in E \mid q_e(\ell_u(\theta)) > 0\}$ denotes the set of **resetting arcs**.

Multi-source single-sink Nash flows over time. A dynamic equilibrium now consists of a feasible flow over time together with an inflow distribution and each particle chooses a convex combination of routes from the sources to the sink such that it arrives there as fast as possible.

Definition 5.11 (Nash flow over time). A tuple $f = ((f_e^+)_{e\in E}, (f_j)_{j\in J})$ consisting of a feasible flow over time and an inflow distribution is a **Nash flow over time** if the following two **Nash flow conditions** hold:

$$\begin{aligned} &\ell_{s_j}(\phi) = T_j(\phi) && \text{for all } j \in J \text{ and almost all } \phi \in \mathbb{R}_{\geq 0}, && \text{(msN1)} \\ &f_e^+(\theta) > 0 \;\Rightarrow\; \theta \in \ell_u(\Phi_e) && \text{for all arcs } e = uv \in E \text{ and almost all } \theta \in [0,\infty), && \text{(msN2)} \end{aligned}$$

where $\Phi_e := \{\, \phi \in \mathbb{R}_{\geq 0} \mid e \in E'_\phi \,\}$ is the set of flow particles for which arc e is active.

Figuratively speaking, these two conditions mean that entering the network through a source s_j is always a fastest way to reach s_j (msN1) and that a Nash flow over time uses only active arcs (msN2), and therefore only shortest paths, to t.

Lemma 5.12. *A tuple $f = ((f_e^+)_{e\in E}, (f_j)_{j\in J})$ of a feasible flow over time and an inflow distribution is a Nash flow over time if, and only if, we have*

$$F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi)) \quad \textit{and} \quad F_j(\phi) = \ell_{s_j}(\phi) \cdot r_j$$

for all arcs $e = uv \in E$, every $j \in J$, and all particles $\phi \in \mathbb{R}_{\geq 0}$.

Proof. From Lemma 3.3 (i) $\Leftrightarrow$ (iii) it follows immediately that (msN2) is satisfied if, and only if, $F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi))$ for all $e \in E$ and all $\phi \in \mathbb{R}_{\geq 0}$. Hence, the lemma is true since $T_j(\phi) = \frac{F_j(\phi)}{r_j}$, and therefore, (msN1) is satisfied if, and only if, $F_j(\phi) = \ell_{s_j}(\phi) \cdot r_j$ for all $j \in J$ and all $\phi \in \mathbb{R}_{\geq 0}$. $\square$

Underlying static flows. The underlying static flow is now given by two types of functions

$$x_e(\phi) := F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi)) \quad \text{and} \quad x_j(\phi) := F_j(\phi) = \ell_{s_j}(\phi) \cdot r_j.$$

For every ϕ this is a static S-t-flow with $x_j(\phi)$ as supply at source s_j since the integral of (5.7) over $[0, \ell_v(\phi)]$ yields

$$\sum_{e \in \delta_v^+} x_e(\phi) - \sum_{e \in \delta_v^-} x_e(\phi) = \begin{cases} 0 & \text{if } v \in V \setminus (S \cup \{t\}), \\ \ell_{s_j}(\phi) \cdot r_j = x_j(\phi) & \text{if } v = s_j \in S. \end{cases} \tag{5.9}$$

Let x'_e, x'_j and ℓ'_v denote the derivative functions, which exist almost everywhere, since the x- and ℓ-functions are monotone. Then, it is possible to determine the inflow function of every arc $e = uv$ as well as the inflow distribution from these derivatives, since

$$x'_e(\phi) = f_e^+(\ell_u(\phi)) \cdot \ell'_u(\phi) \quad \text{and} \quad f_j(\phi) = \ell'_{s_j}(\phi) \cdot r_j.$$

Consequently, a Nash flow over time is, again, completely characterized by these derivatives. Differentiating (5.9) yields that $x'(\phi)$ also forms a static S-t-flow, which we consider next.

Thin flows with resetting for multiple sources and a single sink. Let $E' \subseteq E$ be a subset of arcs such that the subgraph $G' = (V, E')$ is acyclic and every node is reachable by some source within G'. Note that not every node needs to be able to reach sink t. Additionally, we consider a subset of resetting arcs $E^* \subseteq E'$. Moreover, let $X\left(E', (x'_j)_{j \in J}\right)$ be the set of all static S-t-flows in G' with supply x'_j at source s_j for $x'_j \geq 0$ and $\sum_{j \in J} x'_j = 1$.

Definition 5.13 (Thin flow with resetting for multiple sources).

A vector $(x'_j)_{j \in J}$ with $x'_j \geq 0$ and $\sum_{j \in J} x'_j = 1$, together with a static flow $(x'_e)_{e \in E} \in X\left(E', (x'_j)_{j \in J}\right)$ and a node labeling $(\ell'_v)_{v \in V}$ is called **thin flow with resetting** on $E^* \subseteq E'$ if

$$\ell'_{s_j} = \frac{x'_j}{r_j} \qquad \text{for all } j \in J, \tag{msTF1}$$

$$\ell'_{s_j} \leq \min_{e = us_j \in E'} \rho_e(\ell'_u, x'_e) \qquad \text{for all } j \in J, \tag{msTF2}$$

$$\ell'_v = \min_{e = uv \in E'} \rho_e(\ell'_u, x'_e) \qquad \text{for all } v \in V \backslash S, \tag{msTF3}$$

$$\ell'_v = \rho_e(\ell'_u, x'_e) \qquad \text{for all } e = uv \in E' \text{ with } x'_e > 0, \tag{msTF4}$$

$$\text{where} \qquad \rho_e(\ell'_u, x'_e) := \begin{cases} \frac{x'_e}{\nu_e} & \text{if } e = uv \in E^*, \\ \max\left\{ \ell'_u, \frac{x'_e}{\nu_e} \right\} & \text{if } e = uv \in E' \backslash E^*. \end{cases}$$

As before we can prove that the derivatives of a Nash flow over time f form a thin flow with resetting almost everywhere.

Theorem 5.14.

For a Nash flow over time $((f_e^+)_{e \in E}, (f_j)_{j \in J})$ the derivative labels $(x'_j(\phi))_{j \in J}$ and $(x'_e(\phi))_{e \in E'_\phi}$ together with $(\ell'_v(\phi))_{v \in V}$ form a thin flow with resetting on E^*_ϕ in the current shortest paths network $G'_\phi = (V, E'_\phi)$ for almost all $\phi \in \mathbb{R}_{\geq 0}$.

Proof. We have that $x'_j(\phi) = f_j(\phi) \geq 0$ for all $j \in J$ and $\sum_{j \in J} x'_j(\phi) = \sum_{j \in J} f_j(\phi) = 1$ for almost all $\phi \in \mathbb{R}_{\geq 0}$. Equation (msTF1) follows immediately from Lemma 5.12 and Equations (msTF2) and (msTF4) can be proven in the exact same way as (TF2) and (TF3) in Theorem 3.6. □

Constructing Nash flows over time. In order to construct a multi-source Nash flow we first show the existence of multi-source thin flows with resetting.

Theorem 5.15.

Consider an acyclic graph $G' = (V, E')$ with sources S, sink t, capacities ν_e and a subset of arcs $E^* \subseteq E'$. Suppose that every node is reachable by some source. Then there exists a thin flow $\big((x'_j)_{j\in J}, (x'_e)_{e\in E}, (\ell'_v)_{v\in V}\big)$ with resetting on E^*.

In order to show this, the proof of Theorem 3.7 needs only be modified slightly to fit the new definition of a thin flow with resetting in the multi-source setting. The key idea is, again, to use a set-valued function in order to apply Kakutani's fixed-point theorem (Theorem 2.9 on page 22). A detailed version of the proof can be found in the appendix on page 92.

A **restricted inflow distribution** on $[0, \phi)$ is a family of functions $f_j : [0, \phi) \to [0, 1]$ that satisfies $\sum_{j\in J} f_j(\varphi) = 1$ for almost all $\varphi \in [0, \phi)$. A **restricted Nash flow over time** on $[0, \phi)$ is a feasible flow over time together with a restricted Nash flow condition that satisfies the Nash flow conditions (msN1) and (msN2) for almost all particles in $[0, \phi)$ and almost all times in $[0, \ell_u(\phi))$. Note that we do not demand that the cumulative source inflow functions F_j of a restricted inflow distribution are unbounded. This would not make any sense as ϕ is always a trivial upper bound.

In the same manner as described in Section 3.5 it is now possible to extend a restricted Nash flow over time by computing a thin flow with resetting and defining

$$\ell_v(\phi + \xi) := \ell_v(\phi) + \xi \cdot \ell'_v \quad \text{and} \quad \begin{aligned} x_e(\phi + \xi) &:= x_e(\phi) + \xi \cdot x'_e, \\ x_j(\phi + \xi) &:= x_j(\phi) + \xi \cdot x'_j, \end{aligned}$$

for all $v \in V$, $e \in E$, $j \in J$, and $\xi \in [0, \alpha]$ for some $\alpha > 0$ that satisfies (3.9) and (3.10).

Based on this we can extend the inflow functions and the inflow distribution, which gives us

$$\begin{aligned} & f^+_e(\theta) := \frac{x'_e}{\ell'_u} && \text{for } \theta \in [\ell_u(\phi), \ell_u(\phi + \alpha)), \\ & f^-_e(\theta) := \frac{x'_e}{\ell'_v} && \text{for } \theta \in [\ell_v(\phi), \ell_v(\phi + \alpha)) \\ \text{and} \quad & f_j(\varphi) := x'_j \cdot r_j && \text{for } \varphi \in [\phi, \phi + \alpha) \end{aligned}$$

for all $e = uv \in E$ and all $j \in J$. Note that in the case of $\ell'_u = 0$ the time interval $[\ell_u(\phi), \ell_u(\phi + \alpha))$ is empty and the same is true for the corresponding time interval if $\ell'_v = 0$. Once more we call this extended flow α**-extension**.

Theorem 5.16.

Given a restricted Nash flow over time $((f^+_e)_{e\in E}, (f_i)_{i=1}^n)$ on $[0, \phi)$ and an $\alpha > 0$ satisfying (3.9) *and* (3.10)*, then the α-extension is a restricted Nash flow over time on $[0, \phi + \alpha)$.*

Proof. Since

$$\sum_{e\in\delta^+_{s_j}} f^+_e(\theta) - \sum_{e\in\delta^-_{s_j}} f^-_e(\theta) = \sum_{e\in\delta^+_{s_j}} \frac{x'_e}{\ell'_{s_j}} - \sum_{e\in\delta^-_{s_j}} \frac{x'_e}{\ell'_{s_j}} = \frac{x'_j}{\ell'_{s_j}} = r_j$$

Lemma 3.9 implies that the α-extension is a feasible flow over time. Furthermore,

$$\ell_{s_j}(\phi + \xi) = \ell_{s_j}(\phi) + \xi \cdot \ell'_{s_j} \overset{\text{(msN1)}}{=} T_j(\phi) + \xi \cdot \frac{x'_j}{r_j} = T_j(\phi) + \xi \cdot T'_j(\phi) = T_j(\phi + \xi),$$

together with the second part of Lemma 3.9 yields that the ℓ-labels satisfy Equation (5.8), i.e., they match the earliest arrival times. In addition, we have $\sum_{j\in J} f_j(\theta) = \sum_{j\in J} x'_j = 1$ for all

$\theta \in [\phi, \phi+\alpha)$, which shows that $(f_j)_{j\in J}$ is a restricted inflow distribution. Finally, Lemma 5.12 yields $F_e^+(\ell_u(\varphi)) = F_e^-(\ell_v(\varphi))$ and $F_j(\varphi) = \ell_{s_j}(\varphi) \cdot r_j$ for all $\varphi \in [0, \phi)$, so for $\xi \in [0, \alpha)$ it holds that

$$F_e^+(\ell_u(\phi+\xi)) = F_e^+(\ell_u(\phi)) + \frac{x'_e}{\ell'_u} \cdot \xi \cdot \ell'_u = F_e^-(\ell_v(\phi)) + \frac{x'_e}{\ell'_v} \cdot \xi \cdot \ell'_v = F_e^-(\ell_v(\phi+\xi)),$$
$$F_j(\phi+\xi) = F_j(\phi) + \xi \cdot x'_j = \ell_{s_j}(\phi) \cdot r_j + \xi \cdot \ell'_{s_j} \cdot r_j = \ell_{s_j}(\phi+\xi) \cdot r_j.$$

Lemma 5.12 implies that the α-extension is a restricted Nash flow over time on $[0, \phi+\alpha)$. □

Finally, we show that this construction leads to a multi-source Nash flow over time.

Theorem 5.17.
There exists a multi-source single-sink Nash flow over time.

Proof. The proof is the same as for Theorem 3.11 in the base model with the only exception that we have to show in addition that all cumulative source inflow functions are unbounded. We start by showing that the earliest arrival time ℓ_t is unbounded. There cannot be an upper bound B on ℓ_t since the flow rate into t is bounded by $N := \sum_{e\in\delta_t^-} \nu_e$ and with the FIFO principle we obtain that no particle $\phi > N \cdot B$ reaches t before time $\frac{\phi}{N} > B$. Next, we show that all ℓ-labels are unbounded. Suppose this is not true. Since every node can reach t there would be an arc $e = uv$, where ℓ_u is bounded and ℓ_v is not. Since T_e is Lipschitz continuous $T_e \circ \ell_u$ would be bounded as well. But this contradicts that $\ell_v(\phi) \le T_e(\ell_u(\phi))$ goes to ∞ for $\phi \to \infty$. Hence, $F_j(\phi) = \ell_{s_j}(\phi) \cdot r_j$ is unbounded for every $j \in J$, which completes the proof. □

Multi-commodity Nash flows over time with common destination. Finally, we want to show that these multi-source single-sink Nash flows over time do indeed correspond to a multi-commodity Nash flow over time where all commodities share the same destination. To do so, consider a multi-source single-sink Nash flow over time $f = (f_e^+, f_e^-)$ as constructed above with a thin flow $(x'(\phi), \ell'(\phi))$ for each particle ϕ. By adding a super source s and a new arc $e_j = ss_j$ carrying a flow of $x'_j(\phi)$ for each $j \in J$, we obtain an s-t-flow of value 1 and by using the flow decomposition theorem (see Theorem 2.1 on page 11) we obtain a path-based formulation $(x'_P)_{P\in\mathcal{P}}$. For every $j \in J$ let $\mathcal{P}_j$ be all s-t-paths that start with the new arc e_j. By assigning all flow on these paths to commodity j we obtain

$$x'_{j,e} := \sum_{\substack{P\in\mathcal{P}_j \\ \text{with } e\in P}} x_P.$$

Setting

$$f_{j,e}^+(\theta) := \frac{x'_{j,e}(\phi)}{\ell'_u(\phi)} \quad \text{for } \theta = \ell_u(\phi) \qquad \text{and} \qquad f_{j,e}^-(\theta) := \frac{x'_{j,e}(\phi)}{\ell'_v(\phi)} \quad \text{for } \theta = \ell_v(\phi)$$

for all $\phi \in \mathbb{R}_{\ge 0}$ provides a multi-commodity Nash flow over time with unlimited inflow rates as we show in the following theorem.

Theorem 5.18.
The family of functions $(f_{j,e}^+, f_{j,e}^-)_{j\in J, e\in E}$ is a multi-commodity Nash flow over time in a network with inflow rates r_j and $I_j = [0, \infty)$ for all $j \in J$.

Proof. Flow conservation (5.2) for every commodity $j \in J$ on every node $v \in V \setminus \{t\}$ follows immediately since

$$\sum_{e\in\delta_v^+} f_{j,e}^+(\theta) - \sum_{e\in\delta_v^-} f_{j,e}^-(\theta) = \sum_{e\in\delta_v^+} \frac{x'_{j,e}}{\ell'_{j,v}} - \sum_{e\in\delta_v^-} \frac{x'_{j,e}}{\ell'_{j,v}} = \begin{cases} 0 & \text{if } v \neq s_i, \\ \frac{x'_j}{\ell'_v} = r_j & \text{if } v = s_j. \end{cases}$$

The condition on the total outflow rate (5.3) holds as we have shown in Theorem 5.16 and Condition (5.4) is satisfied for each arc $e = uv \in E$ since in the case of $f_e^+(\ell_u(\phi)) > 0$ we have that e is active (from the single-commodity perspective), i.e., $\ell_v(\phi) = T_e(l_u(\phi))$, and therefore,

$$f_{j,e}^-(\ell_v(\phi)) = \frac{x'_{j,e}(\phi)}{\ell'_v(\phi)} = \frac{x'_e(\phi)}{\ell'_v(\phi)} \cdot \frac{x'_{j,e}(\phi)}{\ell'_u(\phi)} \cdot \frac{\ell'_u(\phi)}{x'_e(\phi)} = f_e^-(\ell_v(\phi)) \cdot \frac{f_{j,e}^+(\ell_u(\phi))}{f_e^+(\ell_u(\phi))}.$$

For $f_e^+(\ell_u(\phi)) = 0$ we clearly have $x'_{i,e} = 0$, and thus, $f_{i,e}^-(\ell_v(\phi)) = 0$. Note that the function ℓ_v is continuous and unbounded, and therefore, we can found for every $\theta \geq \ell_v(0)$ a $\phi \in \mathbb{R}_{\geq 0}$ with $\theta = \ell_v(\phi)$. For $\theta < \ell_v(0)$ we have $f_{j,e}^+(\theta) = f_{j,e}^-(\theta) = 0$ as no flow has reached e yet. Hence, we have a feasible multi-commodity flow over time.
The multi-commodity Nash flow condition (mcN) follows immediately by Lemma 5.3 and by

$$F_{j,e}^+(\ell_u(\phi)) = \int_0^\phi f_{j,e}^+(\ell_u(\xi)) \cdot \ell'_u(\xi)\,\mathrm{d}\xi = \int_0^\phi x'_{j,e}(\xi)\,\mathrm{d}\xi = \int_0^\phi f_{j,e}^-(\ell_v(\xi)) \cdot \ell'_v(\xi)\,\mathrm{d}\xi = F_{j,e}^-(\ell_v(\phi)),$$

which holds for all $\phi \in \mathbb{R}_{\geq 0}$. □

5.3 Common Origin

In this last section of the multi-commodity chapter we are going to analyze the second special case of multiple commodities with a common origin. Suppose that each commodity has its own sink but all flow starts at the same common source; see left the side of Figure 5.5. In this scenario the commodities matter a lot, since different flow particles within the network might want to reach different sinks. Nonetheless, we show that this special case, once again, can be reduced to the single commodity case by using a super sink construction as it is shown on the right side of Figure 5.5. The commodities can then be reconstructed by using path decompositions of the thin flows.

Extended graphs. In order to construct a Nash flow over time in this setting we add a super sink to the graph. For this let $\nu_{\min} := \min_{e\in E} \nu_e$ be the minimal capacity of the network, $r := \sum_{j\in J} r_j$ the

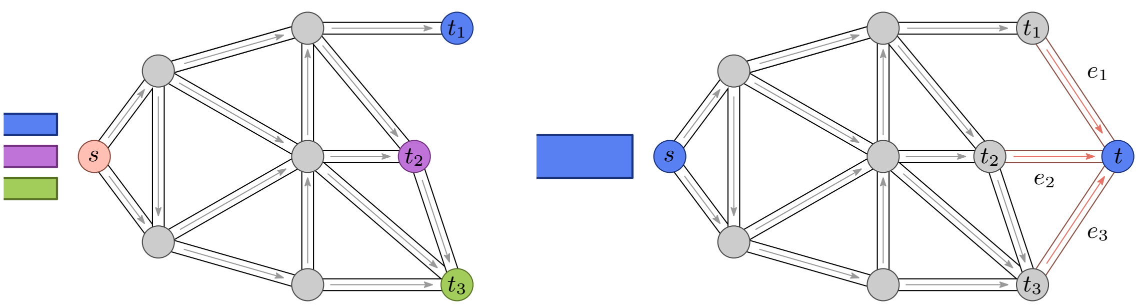


Figure 5.5: *On the left:* A multi-commodity network where all commodities share the same origin. *On the right:* Constructing a multi-commodity Nash flow over time in this setting can be reduced to a single commodity Nash flow over time by adding a super sink t and new arcs e_j with very small capacities.

total network inflow and $\sigma := \min\{\nu_{\min}, r\}$. For all $j \in J$ we define δ_j to be the length of a shortest s-t_j-path according to the transit times. Furthermore, let $\delta_{\max} := \max_{j\in J} \delta_j$ be the maximal distance from the source to a sink. We extend G by a super sink t and $|J|$ new arcs $e_j := (t_j, t)$ with

$$\tau_{e_j} := \delta_{\max} - \delta_j \quad \text{and} \quad \nu_{e_j} := \frac{r_j \cdot \sigma}{2r}. \tag{5.10}$$

We denote the **extended graph** by $\bar{G} := (\bar{V}, \bar{E})$ with $\bar{V} := V \cup \{t\}$ and $\bar{E} := E \cup \{e_1, \dots, e_m\}$. Note that the new capacities are strictly smaller than all original capacities and that they are proportional to the inflow rate of the respective commodity. Furthermore, we choose the transit times such that all new arcs are in the current shortest paths network for particle $\phi = 0$. The reason for the choice of σ is the following.

Lemma 5.19. *For every single-commodity thin flow with resetting (x', ℓ') (see page 33) in $\bar{G}$ it holds for all $v \in V \setminus \{t\}$ that $\ell'_v \le \frac{1}{\sigma}$.*

Proof. We have $\ell'_s = \frac{1}{r} \le \frac{1}{\sigma}$ and $\frac{x'_e}{\nu_e} \le \frac{1}{\nu_{\min}} \le \frac{1}{\sigma}$. Hence, by induction over the acyclic current shortest paths network we obtain that $\ell'_v \le \max\{\frac{x'_e}{\nu_e}, \ell'_u\} \le \frac{1}{\sigma}$ for all active arcs $e = uv$. □

Reduction from single-commodity Nash flows over time. We obtain a multi-commodity Nash flow over time f with common source by using a single-commodity Nash flow over time $\bar{f}$ in $\bar{G}$, which exists due to Theorem 3.11. To prove this we first show that, if all new arcs are active for some particle ϕ, then there is a static flow decomposition of the single-commodity thin flow with resetting x' with $x'_{e_j} = r_j$. This is formalized in the following lemma, where we write $x'|_E$ for the restriction of x' to the original graph G and $|\cdot|$ for the flow value of a static flow.

Lemma 5.20. *For any $E^* \subseteq E' \subseteq \bar{E}$ with $\{e_j \mid j \in J\} \subseteq E'$ consider the single-commodity thin flow with resetting (x', ℓ') with network inflow rate r. There exists a static flow decomposition $(x'_{j,e})_{j\in J, e\in E}$ with $x'|_E = \sum_{j\in J} x'_j$ such that each static flow x'_j conserves flow on all $v \in V \setminus (\{s, t_j\})$ and $|x'_j| = x'_{e_j} = \frac{r_j}{r}$ for $j \in J$.*

Proof. Let $\mathcal{P}$ be the set of all s-t-paths in the current shortest paths network $G' = (V, E')$. Note that G' is always acyclic and x' can therefore be described by the path vector $(x'_P)_{P\in\mathcal{P}}$ due to the flow decomposition theorem (Theorem 2.1 on page 11). For all $j \in J$ let $\mathcal{P}_j$ be the set of all s-t-paths that contain e_j. These sets form a partition of $\mathcal{P}$ since every path has to use exactly one of the new arcs. By setting $x'_j := \sum_{P\in\mathcal{P}_j} x'_P|_E$ we obtain the desired decomposition of x', because $x'_P|_E$ for $P \in \mathcal{P}_j$ conserves flow on all nodes except for the ones in $\{s, t_j\}$ and the same is true for sums of these path flows.

Since x'_j sends $|x'_j|$ flow units from s over e_j to t_j we have $|x'_j| = x'_{e_j}$. It remains to show that $x'_{e_j} = \frac{r_j}{r}$ for all $j \in J$. Suppose that this is not true. Since x' sends exactly $1 = \sum_{j\in J} \frac{r_j}{r}$ flow units from s to t, there has to be an index $a \in J$ with $x'_{e_a} > \frac{r_a}{r}$ and an index $b \in J$ with $x'_{e_b} < \frac{r_b}{r}$.
With Lemma 5.19 it follows that

$$\ell'_{t_b} \le \frac{1}{\sigma} \overset{(5.10)}{<} \frac{r_a}{r \cdot \nu_{e_a}} < \frac{x'_{e_a}}{\nu_{e_a}} \overset{\text{(TF3)}}{\le} \ell'_t \quad \text{and} \quad \frac{x'_{e_b}}{\nu_{e_b}} \overset{(5.10)}{=} \underbrace{\frac{x'_{e_b} \cdot r}{r_b}}_{<1} \cdot \frac{2}{\sigma} < \underbrace{\frac{x'_{e_a} \cdot r}{r_a}}_{>1} \cdot \frac{2}{\sigma} \overset{(5.10)}{=} \frac{x'_{e_a}}{\nu_{e_a}} \overset{\text{(TF3)}}{\le} \ell'_t.$$

But this is a contradiction, since (TF2) yields that $\ell'_t = \min_{j\in J} \rho_{e_j}(\ell'_{t_j}, x'_{e_j})$ and the last two equations show that $\rho_{e_b}(\ell'_{t_b}, x'_{e_b}) < \ell'_t$. Hence, we have $x'_{e_j} = \frac{r_j}{r}$ for all $j \in J$, which finishes the proof. □

As the next step we show that the new arcs are always active.

Lemma 5.21. *In a Nash flow over time $\bar{f}$ in $\bar{G}$ all new arcs $(e_j)_j \in J$ are active for all particles $\phi \in \mathbb{R}_{\geq 0}$.*

Proof. For particle $\phi = 0$ there are no queues yet, and therefore, the exit time for each arc e is $T_e(\theta) = \theta + \tau_e$. Hence, $\ell_{t_j}(0) = \delta_j$ and by construction we have for all $j \in J$ that

$$\ell_t(0) = \ell_{t_j}(0) + \tau_{e_j} = T_{e_j}(\ell_{t_j}(0)).$$

Therefore, all arcs e_j are active at the beginning and also during the first thin flow phase because by Lemma 5.20 we have $x'_{e_j} > 0$ for the first thin flow with resetting, which implies that arc e_j stays active.

Suppose now for contradiction that there are particles for which not all new arcs are active. Let ϕ_0 be the infimum of these particles. By the consideration above we have $\phi_0 > 0$ and Lemmas 5.19 and 5.20 imply that

$$f^+_{e_j}(\ell_{t_j}(\phi)) = \frac{x'_{e_j}}{\ell'_{t_j}} \geq x'_{e_j} \cdot \sigma = \frac{r_j}{r} \cdot \sigma \overset{(5.10)}{>} \nu_{e_j}$$

for almost all $\phi \in [0, \phi_0)$ and all $j \in J$. Hence, Lemma 3.1 (vii) together with the fact that $\ell'_{t_j} > 0$ (due to the positive throughput of x' at t_j) yields

$$\frac{\mathrm{d}}{\mathrm{d}\phi} q_{e_j}(\ell_{t_j}(\phi)) = q'_{e_j}(\ell_{t_j}(\phi)) \cdot \ell'_{t_j}(\phi) > 0.$$

In other words, a queue is building up within $[0, \phi_0)$, and therefore, $q_{e_j}(\ell_{t_j}(\phi_0)) > 0$ for all $j \in J$. The continuity of $q_{e_j} \circ \ell_{t_j}$ implies that there will be positive queues for all $\phi \in [\phi_0, \phi_0 + \varepsilon]$ for sufficiently small $\varepsilon > 0$. Hence, Lemma 3.4 implies that all new arcs are active during this interval contradicting the existence of ϕ_0. $\square$

Nash flows over time decomposition. With the help of the previous lemmas we can decompose the single-commodity Nash flow over time in $\bar{G}$ to obtain a feasible multi-commodity flow over time in the original graph G.

For each thin flow phase $I = [\phi_1, \phi_2)$ with thin flow (x'_e, ℓ'_e) and thin flow decomposition $(x'_j)_{j \in J}$ we set

$$f^+_{j,e}(\theta) := \frac{x'_{j,e}}{\ell'_u} \quad \text{for } \theta \in [\ell_u(\phi_1), \ell_u(\phi_2)) \quad \text{and} \quad f^-_{j,e}(\theta) := \frac{x'_{j,e}}{\ell'_v} \quad \text{for } \theta \in \ell_v(\phi_1), \ell_v(\phi_2))$$

for all $j \in J$ and every $e = uv \in E$. Note that if $\ell'_u = 0$ we have $[\ell_u(\phi_1), \ell_u(\phi_2)) = \emptyset$. We call the family of functions $(f^+_{j,e}, f^-_{j,e})$ a **Nash flow over time decomposition**.

Theorem 5.22.

The Nash flow over time decomposition $(f^+_{j,e}, f^-_{je})$ is a multi-commodity Nash flow over time in the original network.

Proof. Throughout this proof δ^-_v and δ^+_v denote the incoming and outgoing arcs of v within the original network G by. Since the particles are partitioned into thin flow phases we consider each thin flow phase $I = [\phi_1, \phi_2)$ separately.

Let (x', ℓ') be the corresponding thin flow with thin flow decomposition $(x'_j)_{j \in J}$. Furthermore, we denote the interval of local times of particles in I by $I_v := [\ell_v(\phi_1), \ell_v(\phi_2))$ for every node v.

First, we have to show that the flow over time decompositions form a feasible multi-commodity flow over time. For every $j \in J$ and every node $v \in V \setminus \{ t_j \}$ and all $\theta \in I_v$ the flow conservation condition (5.2) holds since

$$\begin{aligned}\sum_{e\in\delta_v^-} f_{j,e}^-(\theta) - \sum_{e\in\delta_v^+} f_{j,e}^+(\theta) &= \sum_{e\in\delta_v^-} \frac{x'_{j,e}}{\ell'_v} - \sum_{e\in\delta_v^+} \frac{x'_{j,e}}{\ell'_v} \\ &= \frac{1}{\ell'_v} \cdot \left(\sum_{e\in\delta_v^-} x'_{j,e} - \sum_{e\in\delta_v^+} x'_{j,e} \right) = \begin{cases} 0 & \text{if } v \in V \setminus \{ s \}, \\ \frac{r_j}{r\cdot \ell'_s} = r_j & \text{if } v = s. \end{cases}\end{aligned}$$

Furthermore, for $x'_e > 0$ we obtain for all $\theta \in I_u$ and $\vartheta = T_e(\theta) \in I_v$ that

$$f_{j,e}^-(\theta) = \frac{x'_{j,e}}{\ell'_v} = \frac{x'_e}{\ell'_v} \cdot \frac{x'^j_e}{\ell'_u} \cdot \frac{\ell'_u}{x'_e} = f_e^-(\theta) \cdot \frac{f_{j,e}^+(\vartheta)}{f_e^+(\vartheta)}$$

and for $x'_e = 0$ we have $x'_{j,e} = 0$, which implies $f_{j,e}^-(\theta) = 0$. This shows that (5.4) is satisfied. Equation (5.3) follows by Lemma 3.9 since the total flow is a feasible flow over time, and therefore, we have a feasible multi-commodity flow over time.
Finally, since

$$F_{j,e}^+(\ell_u(\phi)) = \int_0^\phi f_{j,e}^+(\ell_u(\xi)) \cdot \ell'_u(\xi) \,\mathrm{d}\xi = \int_0^\phi x'^j_e(\xi) \,\mathrm{d}\xi = \int_0^\phi f_{j,e}^-(\ell_v(\xi)) \cdot \ell'_v(\xi) \,\mathrm{d}\xi = F_{j,e}^-(\ell_v(\phi))$$

we obtain by Lemma 5.3 that $(f_{j,e}^+, f_{j_e}^-)$ is indeed a multi-commodity Nash flow over time. □

5.4 Appendix: Technical Proofs

Lemma 5.3. *For a feasible multi-commodity flow over time f the following statements are equivalent:*

(i) f is a multi-commodity Nash flow over time.

(ii) $F^+_{j,e}(\ell_{j,u}(\phi)) = F^-_{j,e}(\ell_{j,v}(\phi))$ for all $e = uv$, $j \in J$ and all $\phi \in \mathbb{R}_{\geq 0}$.

Proof. (i)⇒(ii): Let $\xi \in [0, \phi]$ be maximal with $F^+_{j,e}(\ell_{j,u}(\xi)) = F^-_{j,e}(\ell_{j,v}(\phi))$. Such particle ξ exists due to the intermediate value theorem (Theorem 2.2 on page 16), together with the fact that $F^+_{j,e} \circ \ell_{j,u}$ is continuous and with the following inequality, which follows by the monotonicity of $F^-_{j,e}$ and Lemma 5.1:

$$F^+_{j,e}(\ell_{j,u}(0)) = 0 \quad \leq \quad F^-_{j,e}(\ell_{j,v}(\phi)) \quad \leq \quad F^-_{j,e}(T_e(\ell_{j,u}(\phi))) = F^+_{j,e}(\ell_{j,u}(\phi)).$$

Note that the second inequality holds because of $\ell_{j,v}(\phi) \leq T_{j,e}(\ell_{j,u}(\phi))$. In the case of $\xi = \phi$ we are done, so suppose $\xi < \phi$. For all particles $\varphi \in (\xi, \phi]$ we know that $T_e(\ell_{j,u}(\varphi)) \neq \ell_{j,v}(\phi)$, since otherwise, we had with Lemma 5.1 that $F^+_{j,e}(\ell_{j,u}(\varphi)) = F^-_{j,e}(T_e(\ell_{j,u}(\varphi))) = F^-_{j,e}(\ell_{j,v}(\phi))$, which would contradict the maximality of ξ. Hence, e is not active for particles in $(\xi, \phi]$ which implies $f^+_{j,e}(\theta) = 0$ for almost all $\theta \in \ell_{j,u}((\xi, \phi]) = (\ell_{j,u}(\xi), \ell_{j,u}(\phi)]$ by (mcN). This leads to

$$F^+_{j,e}(\ell_{j,u}(\phi)) - F^-_{j,e}(\ell_{j,v}(\phi)) = F^+_{j,e}(\ell_{j,u}(\phi)) - F^+_{j,e}(\ell_{j,u}(\xi)) = \int_{\ell_{j,u}(\xi)}^{\ell_{j,u}(\phi)} f^+_{j,e}(\vartheta)\,\mathrm{d}\vartheta = 0,$$

which shows (ii).
(ii)⇒(i): Consider a particle ϕ and an arc $e = uv$ such that e is not active for ϕ and j, in other words, $\ell_{j,v}(\phi) < T_e(\ell_{j,u}(\phi))$. Then, the continuity of $\ell_{j,v}$ and $T_e \circ \ell_{j,u}$ implies that there exists an $\varepsilon > 0$ with $\ell_{j,v}(\phi + \varepsilon) < T_e(\ell_{j,u}(\phi - \varepsilon))$ and that e is not active for all particles in $[\phi - \varepsilon, \phi + \varepsilon]$ and j. This, the fact that $f^+_{j,e}$ and $f^-_{j,e}$ are non-negative and Lemma 5.1 gives us

$$\begin{aligned} 0 \leq \int_{\ell_{j,u}(\phi-\varepsilon)}^{\ell_{j,u}(\phi+\varepsilon)} f^+_{j,e}(\xi)\,\mathrm{d}\xi &= \int_{T_e(\ell_{j,u}(\phi-\varepsilon))}^{T_e(\ell_{j,u}(\phi+\varepsilon))} f^-_{j,e}(\xi)\,\mathrm{d}\xi \\ &\leq \int_{\ell_{j,v}(\phi+\varepsilon)}^{T_e(\ell_{j,u}(\phi+\varepsilon))} f^-_{j,e}(\xi)\,\mathrm{d}\xi \\ &= F^-_{j,e}(T_e(\ell_{j,u}(\phi+\varepsilon))) - F^-_{j,e}(\ell_{j,v}(\phi+\varepsilon)) \\ &= F^+_{j,e}(\ell_{j,u}(\phi+\varepsilon)) - F^-_{j,e}(\ell_{j,v}(\phi+\varepsilon)) \\ &\overset{\text{(ii)}}{=} 0. \end{aligned}$$

Hence, $f^+_{j,e}(\theta) = 0$ for almost all $\theta \in [\ell_{j,u}(\phi - \varepsilon), \ell_{j,u}(\phi + \varepsilon)]$. In other words, for almost all $\theta \in [0, \infty)$ it holds that $\theta \notin \ell_{j,u}(\Phi_{j,e}) \Rightarrow f^+_{j,e}(\theta) = 0$. This is true because for $\theta \geq \ell_{j,u}(0)$ we find a particle ϕ with $\ell_{j,u}(\phi) = \theta$, due to the fact that $\ell_{j,u}$ is surjective. This shows that f is a Nash flow over time, which finishes the proof. ☐

Theorem 5.7.

For a multi-commodity Nash flow over time f, the derivatives $(x'_{j,e})_{j\in J, e\in E}$ and $(\ell'_{j,v})_{j\in J, v\in V}$ form a multi-commodity thin flow.

Proof. Let $\phi \in \mathbb{R}_{\geq 0}$ be a particle such that for all arcs $e = uv$ and all $j \in J$ the derivatives of $\ell_{j,u}$, $x_{j,e}$, $y_{j,e}$ and $T_e \circ \ell_{j,u}$ exist. Furthermore, assume that $x'_{j,e}(\phi) = f^+_{j,e}(\ell_{j,u}(\phi)) \cdot \ell'_{j,u}(\phi) = f^-_{j,e}(\ell_{j,v}(\phi)) \cdot \ell'_{j,v}(\phi)$ and that (mcN) as well as the equation in Lemma 5.5 hold. This is given for almost all $\phi \in \mathbb{R}_{\geq 0}$. By (mcN) we have $f^+_{j,e}(\phi) = 0$, and therefore $x'_{i,e}(\phi) = 0$, for all arcs $e \in E \setminus E'_{i,\phi}$, which shows that $(x'_{j,e})_{e\in E}$ is indeed a static flow on $G_{j,\phi}$.
Taking the derivatives of the first equation of (5.6) shows immediately (mcTF1).
In oder to show (mcTF2) we add $f^+_{j,u}(\ell_{j,u}(\phi)) \cdot \ell'_{j,u}(\phi) = x'_{j,e}(\phi)$ to the equation in Lemma 5.5 and obtain

$$f^+_e(\ell_{j,u}(\phi)) \cdot \ell'_{j,u}(\phi) = \sum_{i\in J} f^+_{i,e}(\ell_{j,u}(\phi)) \cdot \ell'_{j,u}(\phi) = x'_{j,e}(\phi) + y'_{j,e}(\phi).$$

Furthermore, by Lemma 3.1 (vii) we have for almost all $\theta \in [0, \infty)$ that

$$T'_e(\theta) = 1 + q'_e(\theta) = \begin{cases} \max\left\{ \frac{f^+_e(\theta)}{\nu_e}, 1 \right\} & \text{if } q_e(\theta) = 0, \\ \frac{f^+_e(\theta)}{\nu_e} & \text{else.} \end{cases}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}\phi} T_e(\ell_{j,u}(\phi)) = T'_e(\ell_{j,u}(\phi)) \cdot \ell'_{j,u}(\phi) = \begin{cases} \max\left\{ \frac{x'_{j,e}(\phi)+y'_{j,e}(\phi)}{\nu_e}, \ell'_{j,u}(\phi) \right\} & \text{if } q_e(\ell_{j,u}(\phi)) = 0, \\ \frac{x'_{j,e}(\phi)+y'_{j,e}(\phi)}{\nu_e} & \text{else.} \end{cases}$$

This, together with (5.6) and the differentiation rule for a minimum (Lemma 2.3), implies (mcTF2).
In oder to prove (mcTF3) suppose $x'_{j,e}(\phi) = f^-_{j,e}(\ell_v(\phi)) \cdot \ell'_{j,v}(\phi) > 0$, which implies $f^+_e(\ell_{j,u}(\phi)) \geq f^+_{j,e}(\ell_{j,u}(\phi)) > 0$. Since e is active for j we have $\ell_{j,v}(\phi) = T_e(\ell_{j,u}(\phi))$. Hence,

$$\begin{aligned}
\ell'_{j,v}(\phi) &= \frac{x'_{j,e}(\phi)}{f^-_{j,e}(\ell_{j,v}(\phi))} \\
&\overset{(5.4)}{=} \frac{x'_{j,e}(\phi) \cdot f^+_e(\ell_{j,u}(\phi))}{f^+_{j,e}(\ell_{j,u}(\phi)) \cdot f^-_e(\ell_{j,v}(\phi))} \\
&= \frac{\ell'_{j,u}(\phi) \cdot f^+_e(\ell_{j,u}(\phi))}{f^-_e(\ell_{j,v}(\phi))} \\
&\overset{(5.3)}{=} \begin{cases} \max\left\{ \ell'_{j,u}(\phi), \dfrac{\ell'_{j,u}(\phi) \cdot f^+_e(\ell_{j,u}(\phi))}{\nu_e} \right\} & \text{if } q_e(\ell_{j,u}(\phi)) = 0, \\ \dfrac{\ell'_{j,u}(\phi) \cdot f^+_e(\ell_{j,u}(\phi))}{\nu_e} & \text{else,} \end{cases} \\
&= \begin{cases} \max\left\{ \ell'_{j,u}(\phi), \dfrac{x'_{j,e}(\phi) + y'_{j,e}(\phi)}{\nu_e} \right\} & \text{if } e \in E'_{j,\phi} \backslash E^*_{j,\phi}, \\ \dfrac{x'_{j,e}(\phi) + y'_{j,e}(\phi)}{\nu_e} & \text{if } e \in E^*_{j,\phi}, \end{cases} \\
&= \rho^{\phi}_{j,e}\left(\ell'_{j,u}(\phi), x'_{j,e}(\phi), y'_{j,e}(\phi)\right).
\end{aligned}$$

Thus, (mcTF3) is fulfilled, which finishes the proof. □

Lemma 5.9. *For every $x' = (x'_{j,e})_{j\in J,e\in E} \in X$ we can construct a vector $(\ell_{j,v})_{j\in J,v\in V}$ of continuous and monotonically increasing functions such that their derivatives $(\ell'_{j,v})_{j\in J,v\in V}$ satisfy* (mcTF1) *and* (mcTF2) *for all $\phi \in [0, H]$, where we define*

$$\phi^i_{j,u}(\phi) := \min\{\, \varphi \geq 0 \mid \ell_{i,u}(\varphi) = \ell_{j,u}(\phi) \,\} \quad \textit{and} \quad y'_{j,e}(\phi) := \sum_{i\in J\setminus\{\,j\,\}} x'_{i,e}(\phi^i_{j,u}(\phi)) \cdot \frac{\ell'_{j,u}(\phi)}{\ell'_{i,u}(\phi^i_{j,u}(\phi))}.$$

Furthermore, the mapping $(x'_{j,e})_{j\in J,e\in E} \mapsto (\ell_{j,u})_{j\in J,u\in V}$ is weak-strong continuous.

Proof. We prove the existence by extending the functions step by step. First, we initialize $\ell_{j,v}(0)$ with the shortest distance from s_j to v, only considering the transit times. For technical reasons we define the ℓ-functions also for negative values by setting $\ell_{j,u}(\phi) := \ell_{j,u}(0) - \frac{\phi}{r_j}$ for all $\phi < 0$. Furthermore, we assume that x' is defined on $\mathbb{R}$ with $x'_{j,e}(\phi) = 0$ for all $\phi \notin [0, H]$.
For the extension step suppose that there is a $\theta_0 \geq 0$ such that each earliest arrival time function $\ell_{j,v}$ is already defined on an interval $(-\infty, \phi_{j,v}]$ with $\ell_{j,v}(\phi_{j,v}) = \theta_0$ and that these ℓ-functions satisfy the condition of the lemma on this interval. Clearly, this is given for $\theta_0 = 0$ as (mcTF1) and (mcTF2) only have to hold for non-negative ϕ.
We are going to extend each function $\ell_{j,v}$ such that the properties hold up to some $\theta_0 + \alpha$.
For $v = s_j$ we set

$$\ell_{j,s_j}(\phi) := \frac{\phi}{r_j}.$$

In order to extend $\ell_{j,v}$ for $v \neq s_j$ we consider all incoming arcs $e = uv \in \delta^-_v$ that are active for $\phi_{j,u}$ according to the function values from the past that are defined already. In other words, we define

$$\delta'_v := \left\{\, e = uv \in \delta^-_v \;\middle|\; \ell_{j,u}(\phi_{j,u}) + \tau_e \leq \ell_{j,v}(\phi_{j,u}) \,\right\}.$$

Since we consider strictly positive transit times this implies $\ell_{j,v}(\phi_{j,u}) \geq \ell_{j,u}(\phi_{j,u}) + \tau_e = \theta_0 + \tau_e > \theta_0$, and hence, $\phi_{j,v} < \phi_{j,u}$, for all $e = uv \in \delta'_v$.
We define for all $\phi \in [\phi_{j,v}, \phi_{j,u})$

$$\phi^i_{j,u}(\phi) := \min\{\, \varphi \geq 0 \mid \ell_{i,u}(\varphi) = \ell_{j,u}(\phi) \,\} \quad \text{and} \quad y'_{j,e}(\phi) := \sum_{i\in J\setminus\{\,j\,\}} x'_{i,e}(\phi^i_{j,u}(\phi)) \cdot \frac{\ell'_{j,u}(\phi)}{\ell'_{i,u}(\phi^i_{j,u}(\phi))}.$$

Here, $\ell'_{j,u}$ and $\ell'_{i,u}$ are the derivatives of the corresponding functions $\ell_{j,u}$ and $\ell_{i,u}$, which are well-defined on $(-\infty, \phi_{j,u})$ and $(-\infty, \phi^i_{j,u}(\phi_{j,u}))$, respectively, as $\ell_{i,u}(\phi^i_{j,u}(\phi_{j,u})) = \ell_{j,u}(\phi_{j,u}) = \theta_0$.
To determine the earliest arrival time of ϕ at v when using arc $e = uv$ we define

$$\rho_{j,e}(\phi) := \begin{cases} \frac{x'_{j,e}(\phi) + y'_{j,e}(\phi)}{\nu_e} & \text{if } \ell_{j,u}(\phi) + \tau_e < \ell_{j,v}(\phi), \\ \max\left\{\, \ell'_{j,u}(\phi), \frac{x'_{j,e}(\phi) + y'_{j,e}(\phi)}{\nu_e} \,\right\} & \text{else.} \end{cases}$$

Finally, we extend the earliest arrival time $\ell_{j,v}$ for $\phi \in (\phi_{j,v}, \phi_{j,v} + \varepsilon]$ by

$$\ell_{j,v}(\phi) := \ell_{j,v}(\phi_{j,v}) + \min_{e\in\delta'_v} \int_{\phi_{j,v}}^{\phi} \rho_{j,e}(\xi)\,\mathrm{d}\xi.$$

If we choose ε to be small enough, such that $\phi_{j,v} + \varepsilon \leq \phi_{j,u}$ for all u with $uv \in \delta'_v$, the right side is always well-defined. Clearly, the extended function $\ell_{j,v}$ is continuous and monotonically increasing, and by construction it satisfies (mcTF2), since an active arc has a positive waiting time at $\ell_{j,u}(\phi)$ if, and only if, $\ell_{j,u}(\phi) + \tau_e < \ell_{j,v}(\phi)$.

Note that all $\ell'_{j,v}$ are bounded from above, as $x'_{j,e}$ is bounded, and therefore, there exists an $\alpha > 0$ independent of θ_0 such that we can extend all ℓ-functions to $\theta_0 + \alpha$. By iteratively applying this extension step we end up with ℓ-functions that at least defined on $[0, H]$.
As the procedure only depends on the x-functions this construction provides a mapping $x' \mapsto \ell$, which is weak-strong continuous as the integration operator on compact intervals is weak-strong continuous in L^2 as we have shown in Lemma 2.7 on page 21. Furthermore, all operations we used, such as taking sums, minima, maxima and doing time-shifting are continuous mappings when considering the L^2-norm. As $x' \to \ell$ is a composition of continuous functions with a weak-strong continuous function it is also weak-strong continuous. □

Theorem 5.15.

Consider an acyclic graph $G' = (V, E')$ with sources S, sink t, capacities ν_e and a subset of arcs $E^* \subseteq E'$. Suppose that every node is reachable by some source. Then there exists a thin flow $\big((x'_j)_{j\in J}, (x'_e)_{e\in E}, (\ell'_v)_{v\in V}\big)$ with resetting on E^*.

Proof. We consider the following compact, convex, and non-empty set

$$X := \left\{ \big((x'_j)_{j\in J}, (x'_e)_{e\in E}\big) \;\middle|\; x'_j \geq 0, \quad \sum_{j\in J} x'_j = 1, \quad (x'_e)_{e\in E} \in X\left(E', (x'_j)_{j\in J}\right) \right\}$$

and the set-valued map $\Gamma\colon X \to 2^X$ defined by

$$x' \mapsto \left\{ y' \in X \;\middle|\; \begin{array}{ll} y'_j = 0 & \text{for all } j \in J \text{ with } \ell'_{s_j} < \frac{x'_j}{r_j}, \\ y'_e = 0 & \text{for all } e = uv \in E' \text{ with } \ell'_v < \rho_e(\ell'_u, x'_e) \end{array} \right\}.$$

Here, $(\ell'_v)_{v\in V}$ are the node labels associated with x' given by the following equations

$$\begin{aligned} \ell'_{s_j} &= \min\left(\left\{ \tfrac{x'_j}{r_j} \right\} \cup \{\, \rho_e(\ell'_u, x'_e) \mid e = us_j \in E' \,\}\right) && \text{for } j \in J, \\ \ell'_v &= \min_{e=uv\in E'} \rho_e(\ell'_u, x'_e) && \text{for } v \in V \setminus S, \end{aligned}$$

which are uniquely defined due to the fact that G' is acyclic. In order to use Kakutani's fixed point theorem, as it is denoted in Theorem 2.9 on page 22, we prove that all conditions are satisfied:

- The set $\Gamma(x')$ is non-empty, because if we consider exactly the sources with $\ell'_{s_j} = \frac{x'_j}{r_j}$ and the arcs $e = uv$ with $\ell'_v = \rho_e(\ell'_u, x'_e)$, then there has to be at least one path P from such a source s_j to the sink t. If we set $y'_j = 1$ and $y'_e = 1$ for all arcs e on P and every other value to 0 we obtain an element in $\Gamma(x')$.

- Clearly, $\Gamma(x')$ is convex since the sources and arcs that can be used for sending flow are fixed within the set.

- In order to show that $\{\, (x', y') \mid y' \in \Gamma(x) \,\}$ is closed let $(x^n, y^n)_{n\in\mathbb{N}}$ be a sequence within this set, i.e., $y^n \in \Gamma(x^n)$. Since both sequences, $(x^n)_{n\in\mathbb{N}}$ and $(y^n)_{n\in\mathbb{N}}$, are contained in the compact set X they both have a limit x^* and y^* within X. Let $(\ell^n)_{n\in\mathbb{N}}$ be the sequence of associated node labels of (x^n) and ℓ^* the node label of x^*. Note that the mapping $x' \mapsto \ell'$ is continuous, and therefore, it holds that $\ell^* = \lim_{n\to\infty} \ell^n$.

  We now prove that $y^* \in \Gamma(x^*)$. Suppose for contradiction that there is a $j \in J$ with $y^*_j > 0$ and $\ell^*_{s_j} < \frac{x^*_j}{r_j}$. Then there has to be an $n_0 \in \mathbb{N}$ with $y^n_j > 0$ and $\ell^n_{s_j} < \frac{x^n_j}{r_j}$ for all $n \geq n_0$. But this is a contradiction to $y^n \in \Gamma(x^n)$. Now suppose that there is an arc $e = uv \in E'$ with $y^*_e > 0$ and $\ell^*_v < \rho_e(\ell^*_u, x^*_e)$. But again, since ρ_e is continuous there has to be an $n_0 \in \mathbb{N}$ such that $y^n_e > 0$ and $\ell^n_v < \rho_e(\ell^n_u, x^n_e)$ for all $n \geq n_0$. Hence, $\{ (x', y') \mid y' \in \Gamma(x) \}$ is closed.

Since all conditions for Kakutani's fixed point theorem are satisfied, there has to be a fixed point x^* of Γ. Let ℓ^* be the corresponding node labeling. We show that it satisfies the thin flow conditions (msTF1) to (msTF4). If we have $x^*_j > 0$, then $\ell^*_{s_j} = \frac{x^*_j}{r_j}$ follows from $x^* \in \Gamma(x^*)$ and if $x^*_j = 0$ it holds that $0 \leq \ell^*_{s_j} \leq \frac{x^*_j}{r_j} = 0$, and therefore, we have equality in both cases, which yields (msTF1). Equations (msTF2) and (msTF3) are satisfied by the construction of ℓ^*. Finally, for every arc $e = uv \in E'$ with $x^*_e > 0$ it holds that $\ell^*_v = \rho_e(\ell^*_u, x^*_e)$ since $x^* \in \Gamma(x^*)$, which shows (msTF4). This shows that x^* together with ℓ^* forms a thin flow with resetting which completes the proof. □

6 Spillback and Kinematic Waves

In the base model the deterministic queuing allows for arbitrarily long queues and congestions will never expand over multiple road segments. This is of course a huge drawback when considering real-world scenarios since long traffic jams are a huge problem in highly congested networks and, in general, they are very important to take into consideration for road planning.

Spillback. As a first aspect, we extend the base model by **spillback**, an effect that can be observed in daily traffic situations. For example, spillback occurs on a highway, when a bottleneck causes a long traffic jam that blocks exits upstream, or during rush hour in a big city where a crossing is impassable due to the congestion on an intersecting road. We incorporate this into the flow over time model by adding a storage capacity to each arc. This way, arcs can become full leading to a reduction of the inflow rate, and thus, causing queues on preceding arcs to grow.

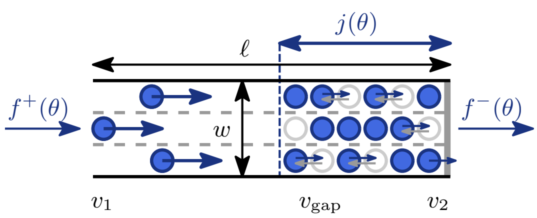
Kinematic waves. Furthermore, traffic congestions are observed to move upstream after a bottleneck has been removed, as vehicles need a certain reaction time to close the gap when a preceding car accelerates. In other words, the gaps between vehicles move backwards over time, which causes a wave-like motion of the congestion. This phenomenon is therefore called **kinematic wave** and it was first studied from a mathematical perspective by Lighthill and Whitham [63, 64]. In order to capture this behavior, we model the gaps between the vehicles by an additional flow over time that travels in the inverse direction and also occupies space on each arc. We will show in an example that this enables us to create the typical real-world phenomenon, where a traffic jam on a highway slowly travels upstream.

It is no surprise that spillback and kinematic waves are of great interest for traffic planners and that these are core features of recent traffic simulation tools. Hence, introducing spillback and kinematic waves is an important step towards closing the gap between mathematical models and simulations. The work presented in this chapter was developed in collaboration with Laura Vargas Koch and the spillback aspect was presented at *ACM-SIAM Symposium on Discrete Algorithms (SODA19)* [87].

6.1 Modeling Kinematic Wave Road Networks

In order to give an intuitive idea of the feature extensions we discuss in this chapter, we extend the basic road model from Section 3.1.

Road model. We consider, once more, a one way road segment (depicted on the left side of Figure 6.1) with a length ℓ, a width or number of lanes w as well as a speed limit v_1. Furthermore, we consider a maximal exit speed v_2, with which vehicles leave the segment when the succeeding roads are free. Note that v_2 might be smaller than the speed limit v_1 due to the geometry of succeeding roads, for example intersection with give way signs or tight curves. Finally, we have a gap speed v_{gaps} that describes how fast the gaps between vehicles travel upstream if the road is congested. Traffic scientists often assume this speed to be globally constant at about $15\,\frac{\text{km}}{\text{h}}$ [24]. Similar to the base model the incoming and outgoing traffic at time θ is denoted by $f^+(\theta)$ and $f^-(\theta)$ and the length of the traffic jam by $j(\theta)$. Note that this time the traffic jam length might reach the road segment length, which means that the road is fully congested. Furthermore, the width w



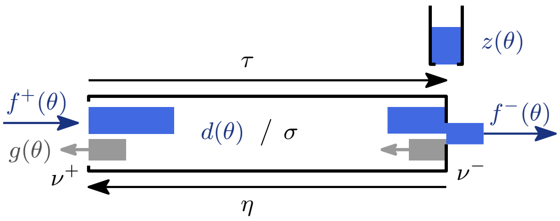


Figure 6.1: *On the left:* A road segment of length ℓ, width w, inflow speed limit v_1, outflow speed limit v_2 and backwards gap speed v_{gap}. Atomic vehicles enter over time (given by $f^+(\theta)$) and queue up in a traffic jam of length $j(\theta)$ before leaving the arc (denoted by $f^-(\theta)$). *On the right:* An arc in the flow over time model with spillback and kinematic waves. We consider a transit time θ, a storage capacity σ, in- and outflow capacities ν^+ and ν^- as well as a gap transit time η. A flow over time is represented by the in- and outflow rates $f^+(\theta), f^-(\theta)$, the queue size $z(\theta)$, the arc load $d(\theta)$ and the gap outflow rate $g(\theta)$ for all times $\theta \in [0, \infty)$.

together with the speed limits v_1 and v_2 induces an upper limit on the incoming and outgoing traffic, i.e., an in- and outflow capacity.

After entering, vehicles first traverse the segment with the maximal speed of v_1 until they reach the end of the traffic jam where they wait in line and only move forward with a speed of v_{gap} when there is enough free space in front of them. When reaching the end of the segment the road users first in line can enter the next road segment provided that it is either not full and still has inflow capacity left, or it is full but has a gap at the very beginning. Note that a road is full when the cars in the traffic jam together with the gaps occupy the whole segment, or in other words, when $j(\theta) = \ell$.

In this vague model congestions can occupy multiple segments and might even block preceding intersections. In addition, the free spaces, created by leaving vehicles, need time to traverse the road in the inverse direction and new cars can only enter a full link if one of these spaces reaches the tail. These are the two key properties for spillback and kinematic waves.

Arc model. In the flow over time model (shown on the right side of Figure 6.1) we equip each arc with a transit time τ corresponding to $\frac{v_1}{\ell}$, an in- and outflow capacity ν^+ and ν^- corresponding to $w \cdot v_1$ and $w \cdot v_2$, a storage capacity σ corresponding to $\ell \cdot w$ and a gap transit time η corresponding to $\frac{v_{\text{gap}}}{\ell}$. The flow is represented by an inflow rate $f^+(\theta)$, an outflow rate $f^-(\theta)$ and a point queue $z(\theta)$. The gaps are modeled by a flow over time traversing upstream denoted by a gap outflow rate of $g(\theta)$. The load $d(\theta)$ is the total amount of flow and gaps currently located on the arc and the link is full when this value reaches the storage capacity.

As in the base model, particles first traverse the complete arc, which takes τ time, before they line up at the point queue located right before the head of the arc. In the best case, flow at the front of the queue leaves the arc with a rate of ν^-. But if the inflow rate of a successive arc is exceeded, the outflow rate might be throttled to match this restriction. At every point in time the leaving flow creates a gap flow that traverses from the head to the tail of the arc and occupies space as well. If an arc gets full, i.e., if $d(\theta) = \sigma$, the inflow rate is further restricted and cannot exceed the gap outflow rate at this point in time. In other words, particles can only enter the arc if there is enough free space (enough gaps) arriving at the entrance.

6.2 Flow Dynamics for Spillback and Kinematic Waves

In this section we properly define the flow over time model that implements spillback and kinematic waves based on the intuition given above. As the spillback aspect is essential for modeling kinematic waves, we simply call it **kinematic wave model**. Note that this is a generalization of the base model as it is always possible to disable both features by setting the inflow capacity large enough and the storage capacity to ∞. To obtain a model that only implements spillback but not kinematic waves it is possible to choose the gap transit time as 0.

Networks. We consider a network consisting of a directed graph $G = (V, E)$ with a source s and a sink t such that each node is reachable by s. At s we have given a **network push rate** of $R \in (0, \infty)$ and every arc $e \in E$ is equipped with a transit time $\tau_e \in [0, \infty)$, an inflow capacity $\nu_e^+ \in (0, \infty)$, an outflow capacity $\nu_e^- \in (0, \infty)$, a storage capacity $\sigma_e \in (0, \infty]$ as well as a gap transit time $\eta_e \in [0, \infty)$. To ensure that the sum of traversing flow and traversing gaps can never fill up an arc on its own and that for full arcs at least some positive gap flow arrives at the tail, we require the following lower bound on the storage capacity of every arc e.

$$\sigma_e > \nu_e^+ \cdot \tau_e + \max \{ \nu_e^+, \nu_e^- \} \cdot \eta_e. \tag{6.1}$$

Furthermore, we require the total transit time of each directed cycle to be strictly positive.

Flow over time. As before, a **flow over time** is a family of locally integrable and bounded functions $f = (f_e^+, f_e^-)_{e \in E}$ that **conserves flow** on all arcs e:

$$F_e^-(\theta + \tau_e) \leq F_e^+(\theta) \qquad \text{for all } \theta \in [0, \infty) \tag{6.2}$$

and at all nodes $v \in V \setminus \{ s, t \}$:

$$\sum_{e \in \delta_v^+} f_e^+(\theta) - \sum_{e \in \delta_v^-} f_e^-(\theta) = 0.$$

Note that this time the **network inflow rate** $r(\theta)$ might be reduced due to spillback, and therefore, we only require

$$r(\theta) \coloneqq \sum_{e \in \delta_s^+} f_e^+(\theta) - \sum_{e \in \delta_s^-} f_e^-(\theta) \in (0, R].$$

Queues, gap flows and arc loads. The **queue** at time θ is, once again, given by

$$z_e(\theta) \coloneqq F_e^+(\theta - \tau_e) - F_e^-(\theta).$$

This time however, a **gap flow** g_e is created whenever flow leaves the arc. Thereby, g_e has the same rate as the outflow rate f_e^- but it travels backwards from the head to the tail in η_e time. We measure this gap flow by the rate of gaps $g_e(\theta)$ that reach the tail. Hence, for all $\theta \geq \eta_e$ we have

$$g_e(\theta) \coloneqq f_e^-(\theta - \eta_e)$$

and for $\theta < \eta_e$ we set $g_e(\theta) \coloneqq 0$. The **total volume of gaps** on an arc e is therefore given by

$$G_e(\theta) \coloneqq \int_\theta^{\theta + \eta_e} g_e(\xi) \,\mathrm{d}\xi.$$

The **arc load** d_e of an arc e at time θ is the total amount of flow on the arc, traversing or in the queue, plus the total volume of gaps in the arc. More precisely, we have

$$d_e(\theta) \coloneqq F_e^+(\theta) - F_e^-(\theta) + G_e(\theta).$$

We say an arc e is **full** at time θ if the load reaches the storage capacity, i.e., if $d_e(\theta) \geq \sigma_e$. However, we will show later on that the arc load can never exceed the storage capacity in a feasible flow over time.

Flow bounds. If there is enough remaining space on an arc the inflow is only bounded by the inflow capacity ν_e^+, but whenever an arc is full we only allow an inflow rate which is at most the gap rate reaching the tail of the arc. Hence, we define the **inflow bound** by

$$b_e^+(\theta) := \begin{cases} \min\{\, g_e(\theta), \nu_e^+ \,\} & \text{if } e \text{ is full at time } \theta, \\ \nu_e^+ & \text{else.} \end{cases}$$

Furthermore, the **push rate** is the flow rate, with which the flow would leave arc e at time θ if it was not restricted by succeeding arcs. It is defined by

$$b_e^-(\theta) := \begin{cases} \nu_e^- & \text{if } z_e(\theta) > 0, \\ \min\{\, f_e^+(\theta - \tau_e), \nu_e^- \,\} & \text{else.} \end{cases}$$

Due to spillback, however, it can happen that the actual outflow rate $f_e^-(\theta)$ is strictly smaller than the push rate $b_e^-(\theta)$. In this case we call arc e **throttled** at time θ.

Feasibility conditions and spillback factor. In order to be **feasible** a flow over time f must satisfy the following four conditions:

- **Inflow condition:** For all $e \in E$ and all $\theta \in [0, \infty)$ we have

$$f_e^+(\theta) \leq b_e^+(\theta).$$

- **Fair allocation condition:** For every node v and all times $\theta \in [0, \infty)$ there exists a $c \in (0, 1]$ such that for all incoming arcs $e \in \delta_v^-$ we have

$$f_e^-(\theta) = \min\{\, b_e^-(\theta), \nu_e^- \cdot c \,\}.$$

  For $v = s$ the network inflow rate must, additionally, satisfy $r(\theta) = R \cdot c$.

- **No-slack condition:** For every node v and for all times $\theta \in [0, \infty)$ we have that if there is at least one incoming arc that is throttled at time θ then there has to be at least one outgoing arc $e \in \delta_v^+$ that satisfies $f_e^+(\theta) = b_e^+(\theta)$.

- **No-deadlock condition:** At any point in time θ the set of arcs with $\eta_e = 0$ that are full at time θ must be cycle free.

For every node v and every time θ we call the maximal value $c \in (0, 1]$ that satisfies the fair allocation condition the **spillback factor** of node v at time θ denoted by $c_v(\theta)$.
In the following we want to give some intuition for all four feasibility conditions. The purpose of the inflow condition is pretty clear. As discussed above, in order to model spillback the inflow rate needs to be restricted not only by the inflow capacity but also by the gap flow in the case that the arc is full. The intuitive idea behind the fair allocation condition is the following. Whenever the total outgoing flow of node v (sum of all inflow rates of all outgoing arcs) is restricted, for example, because a succeeding arc is full, all incoming flow of v needs to merge. To obtain a fair allocation we allow an outflow rate of each incoming arc proportional to its outflow capacity. This time a link with large

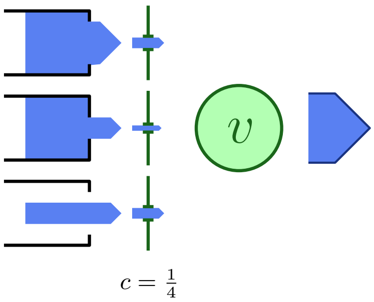


Figure 6.2: In the case that the total push rate exceeds the total outgoing flow of a node v we need to throttle the incoming arcs to preserve flow conservation. For this we determine a value c such that the fair allocation condition is satisfied. We imagine valves after each incoming arc that are shut at the beginning. By gradually increasing c from 0 to 1 we open the valves proportionally to the outflow capacities, i.e., the width of the valve of e equals $c \cdot \nu_e^-$. We stop at the maximal value such that the total incoming flow through the valves matches the total outgoing flow. This number is called spillback factor and in this example we have $c_v(\theta) = \frac{3}{4}$.

outflow capacity can send more flow into v than a link with small outflow capacity. Figuratively speaking, one can imagine a valve right behind the bottleneck of each incoming arc as it is depicted in Figure 6.2. For $c = 0$ these valves are completely closed. By increasing c in a continuous way we open the valves proportionally to the outflow capacities until the total incoming flow of v matches the total outflow of v, such that the flow conservation holds. For $c = 1$ the outflow of each arc in δ_v^- is not restricted at all, and therefore, it matches the push rate.

The reason for the no-slack condition is simple. If some incoming arc e of some node v is throttled (which can only happen if $c_v(\theta) < 1$) then there has to be a reason for this. The only possible reason is, that there is an outgoing arc for which the amount of flow that desires to enter this arc exceeds the inflow bound. Hence, this happens exactly when we have spillback. But whenever there is no restriction on any outgoing arc, no incoming arc should be throttled.

Finally, the no-deadlock condition has a more technical reason. To ensure that the spillback factor $c_v(\theta)$ is always well-defined and strictly positive the inflow bounds $b_{e_1}^+$ of all outgoing arcs $e_1 \in \delta_v^+$ of node v should not directly depend on the outflow rate $f_{e_2}^-(\theta)$ of some incoming arc $e_2 \in \delta_v^-$. To illustrate the issue, consider a directed cycle of, lets say, three arcs e_1, e_2, e_3 that are full at some point in time θ and suppose the gap transit times of all three arcs are 0. If in this setting all particles want to stay in the cycle (and no external flow wants to enter) then the other feasibility conditions would imply that

$$f_{e_1}^+(\theta) = f_{e_1}^-(\theta) = f_{e_2}^+(\theta) = f_{e_2}^-(\theta) = f_{e_3}^+(\theta) = f_{e_3}^-(\theta).$$

This can easily be seen as the inflow condition would imply that the inflow rate $f_{e_1}^+(\theta)$ is smaller or equal to the inflow bound $b_{e_1}^+(\theta)$, which itself is smaller or equal to $g_{e_1}(\theta) = f_{e_1}^-(\theta)$. But by the flow conservation we have $f_{e_1}^-(\theta) = f_{e_2}^+(\theta)$, which again has to be smaller or equal to the inflow bound of e_2. Continuing the argumentation around the cycle shows that all in- and outflow rates have to be equal. Note, however, that this common flow rate is not uniquely determined. In fact, any flow rate strictly larger than 0 but less than the smallest capacity in the cycle (counting in- and outflow capacities) would lead to a feasible flow over time. In principal this would be fine, even though we prefer a unique flow over time for given route choices of the particles. However, as soon as some external flow would try to enter the cycle, everything would come to a complete stop, causing all in- and outflow to be equal to 0. In order to keep the theory as comprehensible as possible we exclude these **strong deadlocks** from the model and require instead the property that, whenever there is a queue on an arc, there has to be some positive outflow rate (see Lemma 6.2).

Note, however, that we allow for **weak deadlocks**. If at least one of the arcs in the cycle has a positive gap transit time, the inflow bound of some arc $e_1 \in \delta_v^+$ does not depend on the outflow rate of an arc $e_2 \in \delta_v^-$ at the very same point in time. This ensures that at least some small flow rate can leave each arc, even though this rate converges to 0 over time. In this scenario we will end up with infinite waiting times, but at least the outflow rate at each arc is always strictly positive. It is worth noting, that even though this extreme case can be handled by the model, for a Nash flow over time neither the hard deadlock nor the soft deadlock ever occur.
The four feasibility conditions above suffice to show the following.

Lemma 6.1. *Given a feasible flow over time f, the following holds for all arcs e and all times θ:*

(i) Outflow capacity condition: $f_e^-(\theta) \leq \nu_e^-$.

(ii) Non-deficit condition: $z_e(\theta) \geq 0$.

(iii) Storage condition: $d_e(\theta) \leq \sigma_e$.

Proof.

(i) The outflow capacity condition follows immediately from the fair allocation condition.

(ii) Assume for contradiction that $z_e(\theta) < 0$ at some point in time θ. Since z_e is continuous there exists an interval $(\theta_0, \theta_1]$ with $z_e(\theta_0) = 0$ and $z_e(\theta) < 0$ for all $\theta \in (\theta_0, \theta_1]$. By the fair allocation condition and the definition of the push rate for the case of $z_e(\theta) \leq 0$ it follows that $f_e^-(\theta) \leq b_e^-(\theta) \leq f_e^+(\theta - \tau_e)$ for all $\theta \in [\theta_0, \theta_1]$. This leads to a contradiction as

$$0 > z_e(\theta_1) - z_e(\theta_0) = \int_{\theta_0}^{\theta_1} f_e^+(\xi - \tau_e) - f_e^-(\xi)\,\mathrm{d}\xi \geq 0.$$

(iii) Assume for contradiction that $d_e(\theta) > \sigma_e$ at some point in time θ. By the continuity of d_e and the fact that $d_e(0) = 0$ it follows that there exists an interval $(\theta_0, \theta]$ with $d_e(\theta_0) = \sigma_e$ and $d_e(\theta) > \sigma_e$ for all $\theta \in (\theta_0, \theta]$. The inflow condition yields that $f_e^+(\theta) \leq g_e(\theta)$ for all $\theta \in [\theta_0, \theta]$. This leads to a contradiction since

$$0 < d_e(\theta) - d_e(\theta_0) = \int_{\theta_0}^{\theta} f_e^+(\xi) - f_e^-(\xi) + g_e(\xi + \eta_e) - g_e(\xi)\,\mathrm{d}\xi = \int_{\theta_0}^{\theta} f_e^+(\xi) - g_e(\xi)\,\mathrm{d}\xi \leq 0.$$

□

Note that (ii) is equivalent to the flow conservation on arcs (6.2). But for the proof we do not use flow conservation on arcs, and therefore the feasibility conditions imply (6.2), exactly as it is the case in the base model.

Congestion suffix. In order to consider gaps in traffic congestions over multiple arcs we need the following definition. A **congestion suffix at time** θ_1 is a path $(e_1, \dots, e_k)$ such that for all $i \in \{1, 2, \dots, k-1\}$ we have that e_i is full at time θ_i with $f_{e_i}^+(\theta_i) = b_{e_i}^+(\theta_i)$ and was throttled at time $\theta_{i+1} := \theta_i - \eta_{e_i}$. Furthermore, arc e_k is not full at time θ_k or was not throttled at time $\theta_k - \eta_{e_k}$, but is also holds that $f_{e_k}^+(\theta_k) = b_{e_k}^+(\theta_k)$.
We can prove that every full arc is part of a congestion suffix, which helps us to show that whenever an arc has a positive queue it also has a positive outflow rate. This property will be important later on. Furthermore, we show that full arcs always have a positive queue.

Lemma 6.2. *For a feasible flow over time f the following statements hold for all $\theta \in [0, \infty)$:*

(i) If e is full at time θ we have $z_e(\theta) > 0$.

(ii) If e is full at time θ we have $z_e(\theta - \eta_e) > 0$.

(iii) Every arc that is full at time θ with $f_e^+(\theta) = b_e^+(\theta)$ is part of a (finite) congestion suffix.

(iv) There is a function $\varepsilon\colon [0, \infty) \to (0, 1)$ depending only on the network but not on f such that every arc e with $z_e(\theta) > 0$ satisfies $f_e^-(\theta) \geq \varepsilon(\theta)$ and $b_e^+(\theta) \geq \varepsilon(\theta)$.

The proofs of (i) and (ii) follow immediately from the lower bound on the storage capacity (6.1) and (iii) can be shown from the definition of congestion suffixes in combination with the no-deadlock condition. Finally, (iv) follows by induction along a congestion suffix. As such suffixes can extend over cycles multiple times, the lower bound on the outflow rate does not only depend on the network but also on the time that has passed since flow has been sent into the network. The formal proofs can be found in the appendix on page 115.

Waiting times. To determine the time a particle spends in the queue we follow a similar approach as introduced for time-dependent capacities in Chapter 4. Note that due to spillback the arc might be throttled during the waiting period, and therefore, the outflow capacity and the queue size alone do not provide the necessary information to determine the waiting time. Instead, we need to consider the actual outflow rate. Hence, for a given feasible flow over time f the waiting time function $q_e\colon [0, \infty) \to [0, \infty]$ of an arc e is defined by

$$q_e(\theta) := \min\left\{ q \geq 0 \;\middle|\; \int_{\theta+\tau_e}^{\theta+\tau_e+q} f_e^-(\xi)\,\mathrm{d}\xi = z_e(\theta + \tau_e) \right\}.$$

Note that it might be the case that $\int_{\theta+\tau_e}^{\infty} f_e^-(\xi)\,\mathrm{d}\xi < z_e(\theta + \tau_e)$. In this case we define $q_e(\theta) := \infty$.

Exit times. Similar to the base model we can now define the exit times $T_e\colon [0, \infty) \to [0, \infty]$ by

$$T_e(\theta) := \theta + \tau_e + q_e(\theta).$$

Note that if $q_e(\theta) = \infty$ the exit time also equals ∞.
With all these definitions we can show next that Lemma 3.1 can be transferred to this kinematic wave model with only some minor changes.

Lemma 6.3. *For a feasible flow over time f it holds for all $e \in E$, $v \in V$ and $\theta \in [0, \infty)$ that:*

(i) $q_e(\theta) > 0 \quad\Leftrightarrow\quad z_e(\theta + \tau_e) > 0$.

(ii) $z_e(\theta + \tau_e + \xi) > 0$ for all $\xi \in [0, q_e(\theta))$.

(iii) $F_e^+(\theta) = F_e^-(T_e(\theta))$ whenever $T_e(\theta) < \infty$.

(iv) For $\theta_1 < \theta_2$ with $F_e^+(\theta_2) - F_e^+(\theta_1) = 0$ and $z_e(\theta_2 + \tau_e) > 0$ we have $T_e(\theta_1) = T_e(\theta_2)$.

(v) If $T_e(\theta) < \infty$ and $f_e^-(T_e(\theta)) = 0$ then $F_e^+(\theta + q_e(\theta)) - F_e^+(\theta) = 0$.

(vi) If $T_e(\theta) < \infty$ the push rate functions satisfy

$$b_e^-(T_e(\theta)) = \begin{cases} \nu_e^- & \text{if } F_e^+(\theta + q_e(\theta)) - F_e^+(\theta) > 0, \\ \min\{\, f_e^+(T_e(\theta) - \tau_e), \nu_e^- \,\} & \text{else.} \end{cases}$$

(vii) *The functions T_e are monotonically increasing.*

(viii) *The functions T_e and q_e are differentiable at almost all θ with $q_e(\theta) < \infty$.*

(ix) *For almost all $\theta \in [0, \infty)$ with $q_e(\theta) < \infty$ we have*

$$q'_e(\theta) = \begin{cases} \frac{f_e^+(\theta)}{f_e^-(T_e(\theta))} - 1 & \text{if } f_e^-(T_e(\theta)) > 0, \\ -1 & \text{else if } z_e(\theta + \tau_e) > 0, \\ 0 & \text{else.} \end{cases}$$

Even though the results are basically the same as the statements of Lemma 3.1, the proofs are more sophisticated in most cases. They can be found with all technical details in the appendix on page 116.

Example. One of the main features we want to capture with the kinematic wave model are the congestion waves that can be observed in real-world traffic. These occur when there is a temporary bottleneck on a heavily frequented highway. The bottleneck creates a traffic jam but even after the bottleneck is removed the congestion persists and moves slowly upstream. This is a phenomenon that cannot be modeled in the base model or a spillback model without kinematic waves. However, as the example in Figure 6.3 shows, this is exactly what happens in the kinematic wave model. Note that arcs downstream remain full, but after some point in time they do not throttle the preceding arc anymore and a big part of the arc load is due to the gap flow. In other words, road users would not experience these arcs as full as there is less traffic and no rate reduction.

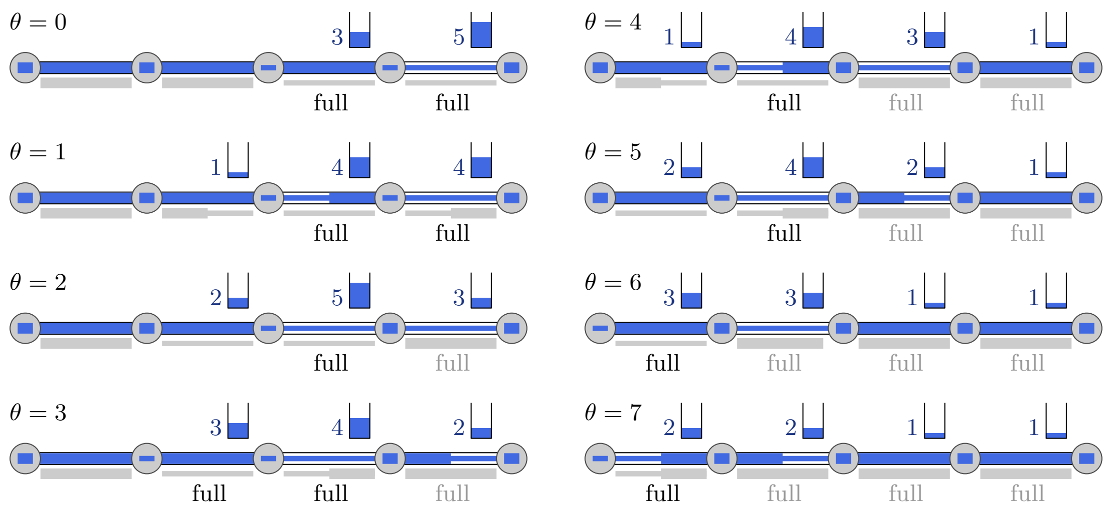


Figure 6.3: Example of a backwards moving wave over time (*fist left column from top to bottom then right column from top to bottom*). An inflow rate of 2 enters a long road that is congested due to some temporary bottleneck (during $\theta < 0$). Every road segment is identical with $\tau_e = \eta_e = \nu^+ = \nu^- = 2$ and $\sigma_e = 9$. Thick blue lines correspond to a flow rate of 2 and thin blue lines to a rate of 1. Within each node we display the throughput rate from this time onwards up to the next time step. Below each arc the gap flow is visualized. Arcs that are full but do not throttle the preceding arc are labeled with a gray "full". After the bottleneck is removed at time 0 the flow can leave the network with rate 2 but due to the kinematic waves it takes time until the flow rate upstream benefits from this.

6.3 Nash Flows Over Time

In this section we define Nash flows over time in the kinematic wave model and transfer the main results from the base model. We keep the structure very similar to the base version but most of the proofs are much more sophisticated than before.

Earliest arrival times. To determine at which time a particle $\phi \in \mathbb{R}_{\geq 0}$ can arrive at some node v for a given feasible flow over time f we first have to determine the earliest time this particle can enter the network at s. This is given by the **source arrival time** defined by

$$\ell_s(\phi) := \min \left\{ \theta \geq 0 \;\middle|\; \int_0^\theta r(\xi)\,\mathrm{d}\xi = \phi \right\}. \tag{6.3}$$

This is basically the same as if particle ϕ stands in a queue right before the source at time 0 with all particles $[0, \phi)$ in front of it. For all other nodes $v \in V \setminus \{ s \}$ we define the **earliest arrival time** by

$$\ell_v(\phi) := \min_{\substack{e=uv\in\delta_v^-,\\ \ell_u(\phi)<\infty}} T_e(\ell_u(\phi)). \tag{6.4}$$

If the minimum is empty we set $\ell_v(\phi) := \infty$. Clearly, ℓ_v is monotone and almost everywhere differentiable on the set $\{ \phi \in \mathbb{R}_{\geq 0} \mid \ell_v(\phi) < \infty \}$ as it is a minimum of functions with these properties; see Lemma 2.3 and Lemma 6.3 (vii) and (viii).

Active, resetting and spillback arcs. The sets of **active** and **resetting arcs** ar, once more, given by

$$E'_\phi := \{ e = uv \in E \mid \ell_u(\phi) < \infty \text{ and } \ell_v(\phi) = T_e(\ell_u(\phi)) \}$$
$$\text{and} \quad E^*_\phi := \{ e = uv \in E \mid \ell_u(\phi) < \infty \text{ and } q_e(\ell_u(\phi)) > 0 \}.$$

Clearly, the **current shortest paths network** $G'_\phi = (V, E'_\phi)$ is cycle free for every $\phi \in \mathbb{R}_{\geq 0}$, since we require the sum of transit times in each cycle to be strictly positive. Furthermore, we call arcs that are full at $\ell_u(\phi)$ **spillback arcs** denoted by

$$\bar{E}_\phi := \{ e = uv \mid \ell_u(\phi) < \infty \text{ and } d_e(\ell_u(\phi)) = \sigma_e \}.$$

Dynamic equilibria. Again, a dynamic equilibrium is a feasible flow over time where almost every particle chooses a fastest route from s to t. As the active arcs once more indicate which paths are the fastest the definition of Nash flows over time is exactly the same as in the base model.

Definition 6.4 (Nash flow over time in the kinematic wave model).
A feasible flow over time f in the kinematic wave model is a **Nash flow over time**, also called **dynamic equilibrium**, if the following **Nash flow condition** holds:

$$f_e^+(\theta) > 0 \;\Rightarrow\; \theta \in \ell_u(\Phi_e) \qquad \text{for all arcs } e = uv \in E \text{ and almost all } \theta \in [0, \infty), \tag{N}$$

where $\Phi_e := \{ \phi \in \mathbb{R}_{\geq 0} \mid e \in E'_\phi \}$ is the set of flow particles for which arc e is active.

As a first important step, we show that the earliest arrival times in a Nash flow over time will never be ∞, and therefore, user equilibria do not create any deadlocks.

Theorem 6.5.
Given a Nash flow over time in the kinematic wave model, for every $e \in E$ and $v \in V$ we have that $q_e(\theta), T_e(\theta), \ell_v(\theta) < \infty$ for all $\theta \in [0, \infty)$.

Proof. In order to simplify the proof we assume that all outgoing arcs from s have inflow capacities greater or equal to the network push rate R and infinite storage capacities. This ensures that the network inflow rate will never be throttled, i.e., $r(\phi) = R$ for all $\phi \in \mathbb{R}_{\geq 0}$. This assumption is without loss of generality, since we can always modify the network by adding a new source and a new arc $e^* = s^*s$ with $\nu^+_{e^*} = \nu^-_{e^*} = R$, $\tau_{e^*} = \eta_{e^*} = 0$ and $\sigma_{e^*} = \infty$. Nash flows over time will then be exactly the same in the new network and the only difference is that, when we have spillback on the original source s, the flow queues on the new arc e^* instead of outside of the network.
With this assumption we first show that $\ell_v(\phi) < \infty$. Assume for contradiction that there is a first particle ϕ_0 where some ℓ-label equal ∞. Since we have $\ell_s(\phi_0) = \frac{\phi_0}{R}$ by our assumption, there has to be some arc $e = uv$ with $\ell_u(\phi_0) < \infty = \ell_v(\phi_0)$, and thus $q_e(\ell_u(\phi_0)) = \infty$. This is only possible if $f^-_e(\xi) \to 0$ for $\xi \to \infty$. In other words, for some $\xi_0 > 0$ arc e would be throttled for all times $\xi > \xi_0$. Consequently, for each $\xi \geq \xi_0$ there has to be some arc $e' \in \delta^+_v$ that is full due to the no-slack condition. For some $\varepsilon < \min \{ \sigma_{e'} - \nu^-_{e'} \cdot \eta_{e'} \mid e' \in E \}$ we consider the particle $\phi_1 := \phi_0 - \varepsilon$, for which $\ell_t(\phi_1) < \infty$. At time $\ell_t(\phi_1)$ all particles $[0, \phi_1)$ have left the network, and therefore, the amount of flow in the network that is in front of particle ϕ_0 equals ε at this point in time. But this is a contradiction since for every arc $e' \in \delta^+_e$ that is full at time $\ell_t(\phi_1)$ we have

$$\sigma_{e'} = d_{e'}(\ell_t(\phi_1)) \leq \varepsilon + G_{e'}(\ell_t(\phi_1)) < \varepsilon + \nu^-_e \cdot \eta_e < \sigma_{e'}.$$

Hence, $\ell_v(\phi) < \infty$ for all $v \in V$ and all ϕ. It follows that this also holds for $q_e(\theta)$ and $T_e(\theta)$ for all $e \in E$ at all times θ. □

With this, Lemma 3.3 transfers to the kinematic wave model one-to-one with the exact same proof, which can be found in the appendix of Chapter 3 on page 47. In particular, we have that a feasible flow over time f is a Nash flow over time if, and only if,

$$F^+_e(\ell_u(\phi)) = F^-_e(\ell_v(\phi)) \qquad \text{for all arcs } e = uv \text{ and all particles } \phi.$$

The active, resetting and spillback arcs in a Nash flow over time have the following properties.

Lemma 6.6. *Given a Nash flow over time the following holds for all times θ:*

(i) $E^*_\phi \subseteq E'_\phi$.

(ii) $E'_\phi = \{ e = uv \mid \ell_v(\phi) \geq \ell_u(\phi) + \tau_e \}$.

(iii) $E^*_\phi = \{ e = uv \mid \ell_v(\phi) > \ell_u(\phi) + \tau_e \}$.

(iv) $\bar{E}_\phi \subseteq E'_\phi$.

(v) $\ell_u(\phi) < \ell_v(\phi)$ *for all* $e = uv \in \bar{E}_\phi$.

The proofs of (i) to (iii) are exactly the same as for the base model (see Lemma 3.4) and can be found on page 49. The last two statements follow with some elementary calculations and Lemma 6.2. The details can be found in the appendix on page 118.

Underlying static flow. In order to obtain some structural insight into Nash flows over time we define, as in the base model, the **underlying static flow** for particle ϕ by

$$x_e(\phi) := F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi)).$$

For every particle ϕ we have that $(x_e(\phi))_{e\in E}$ is, once more, a static s-t-flow of value ϕ and by taking the derivatives $(x'_e(\phi))_{e\in E}$ we obtain a static s-t-flow of value 1 with

$$x'_e(\phi) = f_e^+(\ell_u(\phi)) \cdot \ell'_u(\phi) = f_e^-(\ell_v(\phi)) \cdot \ell'_v(\phi). \tag{6.5}$$

6.4 Spillback Thin Flows

In order to show the existence of Nash flows over time in the kinematic wave model, we show that the derivatives of the underlying static $(x'_e(\phi))_{e\in E}$, corresponding to the strategy of the particles, together with the node labels $(\ell'_v(\phi))_{v\in V}$ and the newly introduced spillback factors $(c_v(\phi))_{v\in V}$ form a spillback thin flow. For a proper definition consider an acyclic directed graph $G' = (V, E')$ with a source s and a sink t where all nodes are reachable from s. Every arc e is equipped with an outflow capacity $\nu_e^- > 0$ and an inflow bound $b_e^+ > 0$. Additionally, we are given a subset of arcs $E^* \subseteq E'$ and a network push rate R.

Definition 6.7 (Spillback thin flow).

A static s-t-flow $(x'_e)_{e\in E}$ of value 1 together with two node labelings $\ell'_v \geq 0$ and $c_v \in (0,1]$, for all $v \in V$, is a **spillback thin flow** with resetting on E^* if it fulfills the following conditions:

$$\begin{aligned}
\ell'_s &= \frac{1}{c_s \cdot R}, && && \text{(sTF1)}\\
\ell'_v &= \min_{e=uv\in E'} \rho_e(\ell'_u, x'_e, c_v) && \text{for } v \in V\backslash\{\,s\,\}, && \text{(sTF2)}\\
\ell'_v &= \rho_e(\ell'_u, x'_e, c_v) && \text{for } e = uv \in E' \text{ with } x'_e > 0, && \text{(sTF3)}\\
\ell'_v &\geq \max_{e=vw\in E'} \frac{x'_e}{b_e^+} && \text{for } v \in V, && \text{(sTF4)}\\
\ell'_v &= \max_{e=vw\in E'} \frac{x'_e}{b_e^+} && \text{for } v \in V \text{ with } c_v < 1, && \text{(sTF5)}
\end{aligned}$$

where

$$\rho_e(\ell'_u, x'_e, c_v) := \begin{cases} \frac{x'_e}{c_v \cdot \nu_e^-} & \text{if } e = uv \in E^*,\\ \max\left\{\,\ell'_u, \frac{x'_e}{c_v\cdot\nu_e^-}\,\right\} & \text{if } e = uv \in E'\backslash E^*. \end{cases}$$

Note that the inflow bounds b_e^+ are part of the input similar to the outflow capacities ν_e^-. During the construction of a Nash flow over time we will set $b_e^+ = b_e^+(\ell_u(\phi))$ for all $e = uv$.
Similar to the base model, we have Condition (sTF2) to ensure that the node label ℓ'_v corresponds to the slope of the earliest arrival time, Condition (sTF1) to make sure that ℓ'_s denotes the derivative of the source arrival time and Condition (sTF3) to guarantee that flow is only sent along arcs that stay active during the phase. Since the outflow capacities might be reduced due to spillback, we multiply them by the spillback factors c_v, which are additional parameters of the spillback thin flow.
However, we have two new conditions for every node v this time, which set the node label ℓ'_v in correlation to the outgoing arcs $e = vw \in \delta_v^+$. To ensure that the outflow rate $f_e^+(\ell_v(\phi)) = \frac{x'_e(\phi)}{\ell'_v(\phi)}$ stays smaller than the inflow bound $b_e^+(\ell_u(\phi))$ we require $\ell'_v \geq \frac{x'_e}{b_e^+}$ in (sTF4). If spillback occurs,

which implies that $c_v < 1$, Condition (sTF5) provides that the no-slack condition is satisfied, since $l'_v = \frac{x'_e}{b^+_e}$ means that $f^+_e(\ell_v(\phi)) = b^+_e(\ell_v(\phi))$.

We also want to present another perspective: If we only consider (sTF1) to (sTF3) and set all spillback factors c_v to 1, we get exactly a thin flow with resetting in the base model (see Theorem 3.5 on page 33), where the ℓ'_v-labels are uniquely defined (Proposition 3.8 on page 36). So it might happen that the inflow rate exceeds the inflow bound on some arcs. Hence, we introduce (sTF4) as a lower bound on ℓ'_v. But since the labels in the base model are unique and they might not satisfy the added lower bounds, we could not guarantee existence anymore. Introducing the spillback factors c_v as additional parameters enables the thin flow to reduce the effective capacity $(c_v \cdot \nu^-_e)$ on each arc, and therefore, to increase the node label ℓ'_v until it satisfies the lower bound of (sTF4). Finally, to ensure that the effective capacity is not reduced unnecessarily, we include (sTF5) to ensure that if c_v is reduced then only to the point that (sTF4) is tight.

In order to show this intuition mathematically we first show that the derivatives of a Nash flow over time in the kinematic wave model are indeed spillback thin flows.

Theorem 6.8.

For almost all $\phi \in \mathbb{R}_{\geq 0}$ the derivatives $x'_e(\phi)$ and $\ell'_v(\phi)$ of a Nash flow over time f together with the spillback factors $c_v(\ell_v(\phi))$ form a spillback thin flow on the current shortest paths network $G'_\phi = (V, E'_\phi)$ with resetting on E^*_ϕ and inflow bounds $b^+_e(\ell_u(\phi))$.

Proof. We fix a particle ϕ such that for all $e = uv \in E$ the derivatives of x_e, ℓ_v and $T_e \circ \ell_u$ exist and $x'_e(\phi) = f^-_e(\ell_v(\phi)) \cdot \ell'_v(\phi) = f^+_e(\ell_u(\phi)) \cdot \ell'_u(\phi)$. Note that almost all particles satisfy these conditions. For the sake of readability, let $\ell'_v := \ell'_v(\phi)$, $x'_e := x'_e(\phi)$, $c_v := c_v(\ell_v(\phi))$, $b^+_e := b^+_e(\ell_u(\phi))$, $E' := E'_\phi$ and $E^* := E^*_\phi$.

(sTF1) By (6.3) we have that $\int_0^{\ell_s(\phi)} r(\xi)\,\mathrm{d}\xi = \phi$. Taking the derivative yields $r(\ell_s(\phi)) \cdot \ell'_s(\phi) = 1$, and therefore, the fair allocation condition implies

$$\ell'_s(\phi) = \frac{1}{r(\ell_s(\phi))} = \frac{1}{R \cdot c_s(\ell_s(\phi))}.$$

(sTF2) Applying the differentiation rule for a minimum (see Lemma 2.3 on page 17) on (6.4) yields

$$\ell'_v = \min_{e=uv \in E'} T'_e(\ell_u(\phi)) \cdot \ell'_u.$$

Note that E' is precisely the set of arcs with $\ell_v(\phi) = T_e(\ell_u(\phi))$, and therefore, exactly these arcs need to be considered for the derivative. In the following we analyze the derivative of $T_e(\phi) = \phi + \tau_e + q_e(\phi)$ at the point $\ell_u(\phi)$ for active arcs $e = uv \in E'$. Lemma 6.3 (ix) yields

$$T'_e(\ell_u(\phi)) = \begin{cases} \frac{f^+_e(\ell_u(\phi))}{f^-_e(\ell_v(\phi))} & \text{if } f^-_e(\ell_v(\phi)) > 0, \\ 0 & \text{else if } z_e(\ell_u(\phi) + \tau_e) > 0, \\ 1 & \text{else.} \end{cases}$$

First, we consider the case that $f^-_e(\ell_v(\phi)) = 0$, which implies that $x'_e = 0$, and hence,

$$T'_e(\ell_u(\phi)) \cdot \ell'_u = \left\{ \begin{array}{ll} 0 & \text{if } q_e(\ell_u(\phi)) > 0, \\ \ell'_u & \text{else,} \end{array} \right\} = \rho_e(\ell'_u, x'_e, c_v).$$

Next, we consider the case that $f_e^-(\ell_v(\phi)) > 0$ and $x'_e = 0$. If $e \notin E^*$, we have

$$f_e^+(\ell_u(\phi)) = f_e^+(\ell_v(\phi) - \tau_e) \geq b_e^-(\ell_v(\phi)) \geq f_e^-(\ell_v(\phi)) > 0,$$

which implies $\ell'_u = \frac{x'_e}{f_e^+(\ell_u(\phi))} = 0$. In both cases, whether $e \in E^*$ or not, we have

$$T'_e(\ell_u(\phi)) \cdot \ell'_u = \frac{x'_e}{f_e^-(\ell_v(\phi))} = 0 = \rho_e(\ell'_u, x'_e, c_v).$$

Finally, we consider $f_e^-(\ell_v(\phi)) > 0$ and $x'_e > 0$. This implies that $x_e(\phi) = F_e^+(\ell_u(\phi))$ is strictly increasing in $[\ell_u(\phi), \ell_u(\phi) + \varepsilon]$, and therefore $F_e^+(\ell_u(\phi) + q_e(\ell_u(\phi))) - F_e^+(\ell_u(\phi)) > 0$ if, and only if, $q_e(\ell_u(\phi)) > 0$. Together with Lemma 6.3 (vi) we obtain

$$b_e^-(\ell_v(\phi)) = \begin{cases} \nu_e^- & \text{if } e \in E^*, \\ \min\{\, f_e^+(\ell_u(\phi)), \nu_e^- \,\} & \text{if } e \in E' \backslash E^*. \end{cases}$$

Hence,

$$\begin{aligned} T'_e(\ell_u(\phi)) \cdot \ell'_u &= \frac{x'_e}{f_e^-(\ell_v(\phi))} \\ &= \frac{x'_e}{\min\{\, c_v \cdot \nu_e^-, b_e^-(\ell_v(\phi)) \,\}} \\ &= \left\{ \begin{array}{ll} \frac{x'_e}{c_v \cdot \nu_e^-} & \text{if } e \in E^* \\ \max\left\{ \frac{x'_e}{f_e^+(\ell_u(\phi))}, \frac{x'_e}{c_v \cdot \nu_e^-} \right\} & \text{if } e \in E' \backslash E^* \end{array} \right\} \\ &= \rho_e(\ell'_u, x'_e, c_v). \end{aligned} \tag{6.6}$$

In summary, we obtain

$$\ell'_v = \min_{e = uv \in E'} T'_e(\ell_u(\phi)) \cdot \ell'_u = \min_{e = uv \in E'} \rho_e(\ell'_u, x'_e, c_v).$$

(sTF3) For $x'_e = f_e^-(\ell_v(\phi)) \cdot \ell'_v > 0$ we have $\ell'_v = \frac{x'_e}{f_e^-(\ell_v(\phi))} \overset{(6.6)}{=} \rho_e(\ell'_u, x'_e, c_v)$.

(sTF4) By the inflow condition we obtain for all arcs $e = vw$ that $x'_e = f_e^+(\ell_v(\phi)) \cdot \ell'_v \leq b_e^+ \cdot \ell'_v$.

(sTF5) Suppose we have $c_v < 1$. The maximality of c_v implies that there has to be at least one incoming throttled arc and by the no-slack condition there has to be an outgoing arc $e = vw$ with $f_e^+(\ell_v(\phi)) = b_e^+$. Hence, $x'_e = f_e^+(\ell_v(\phi)) \cdot \ell'_v = b_e^+ \cdot \ell'_v$. Together with (sTF4) we obtain (sTF5). □

6.5 Existence of Spillback Thin Flows

In this section we show that for all acyclic current shortest paths networks $G' = (V, E')$ with arbitrary capacities, outflow bounds and resetting arcs E^* there always exists a spillback thin flow. Unfortunately, Kakutani's fixed point theorem does not suffice to show this anymore, but we use a finite dimensional variational inequality and the corresponding nonlinear complementarity problem instead.

Variational inequality and nonlinear complementarity. As described in Section 2.5.1 the **variational inequality problem** $\mathrm{VI}(X, \Gamma)$ for a finite index set I and a given subset $X \subseteq \mathbb{R}^I$ as well as a mapping $\Gamma\colon X \to \mathbb{R}^I$ is the following.

$$\text{Find } x \in X \text{ such that} \quad (y - x)^{\mathsf{T}} \cdot \Gamma(x) \geq 0 \quad \text{for all } y \in X. \tag{VI}$$

The set of points that solves this variational inequality is denoted by $\mathrm{SOL}(X,\Gamma)$ and, in order to show that this set contains at least one element, we have to ensure that X is non-empty, compact and convex and that Γ is continuous, as we can then apply Theorem 2.6 (see page 19).
Recall that whenever $X = \bigtimes_{i\in I}[0, M_i]$ is a box then, given a solution of the variational inequality $x^* \in \mathrm{SOL}(X,\Gamma)$, we have that for every $i \in I$ with $x_i^* < M_i$ the **nonlinear complementarity** holds:

$$\Gamma_i(x^*) \geq 0 \qquad \text{and} \qquad x_i^* \cdot \Gamma_i(x^*) = 0. \tag{NCP}$$

In order to define X and Γ for our purpose let $\bar{V} := \{\, \bar{v} \mid v \in V \,\}$ be a copy of the set of nodes V and let $I := E' \,\dot\cup\, V \,\dot\cup\, \bar{V}$ be the index set. We will later see that the index $e \in E'$ corresponds to x'_e, the index $v \in V$ to ℓ'_v and the index $\bar{v} \in \bar{V}$ to β_v, which itself corresponds bijectively to c_v. With

$$\nu^-_{\min} := \min_{e\in E'} \nu^-_e, \qquad \nu^-_{\max} := \max_{e\in E'} \nu^-_e \quad \text{and} \quad b^+_{\min} = \min_{e\in E'} b^+_e$$

we define

$$M := \max\left\{ 1, \frac{R}{\nu^-_{\min}}, \frac{1}{b_{\min}}, \frac{\nu^-_{\max}\cdot |E'|}{R\cdot b^+_{\min}} \right\}, \tag{6.7}$$

$$X := \left\{ (x',\ell',\beta) \in \mathbb{R}^I \;\middle|\; \begin{array}{ll} 0 \leq x'_e \leq 4M^2\cdot \nu^-_e & \text{for all } e \in E' \\ 0 \leq \ell'_v \leq 3M^2 & \text{for all } v \in V \\ 0 \leq \beta_v \leq \log(2M\cdot R) & \text{for all } \bar{v} \in \bar{V} \end{array} \right\},$$

$$\Gamma_i(x',\ell',\beta) := \begin{cases} \dfrac{x'_e}{e^{-\beta_v}\cdot \nu^-_e} - \ell'_v & \text{if } i = e = uv \in E^*, \\ \max\left\{ \ell'_u, \dfrac{x'_e}{e^{-\beta_v}\cdot\nu^-_e} \right\} - \ell'_v & \text{if } i = e = uv \in E'\backslash E^*, \\ \displaystyle\sum_{e\in\delta^-_v} x'_e - \sum_{e\in\delta^+_v} x'_e & \text{if } i = v \in V\backslash \{\, s,t \,\}, \\ \displaystyle\sum_{e\in\delta^-_t} x'_e - \sum_{e\in\delta^+_t} x'_e - 1 & \text{if } i = t \in V, \\ \ell'_s - \dfrac{1}{e^{-\beta_s}\cdot R} & \text{if } i = s \in V, \\ \ell'_v - \displaystyle\max_{e\in\delta^+_v} \frac{x'_e}{b^+_e} & \text{if } i = \bar{v} \in \bar{V}. \end{cases}$$

Note that the e in $e^{-\beta_s}$ stands for the Euler constant. Even though e is still also used for arcs, it should be very clear from the context whether e denotes the Euler constant.
Since X is convex and compact and Γ is continuous Theorem 2.6 states that there exists a solution $(x',\ell',\beta) \in \mathrm{SOL}(X,\Gamma)$.
In order to show that this solution corresponds to a spillback thin flow we need the nonlinear complementarity property. Therefore, we first show that the components of (x',ℓ',β) do not hit the upper boundaries of the box X.

Lemma 6.9. *For every solution $(x',\ell',\beta) \in \mathrm{SOL}(X,\Gamma)$ we have*

(i) $x'_e < 4M^2\cdot\nu^-_e$ *for every arc e,*

(ii) $\ell'_v < 3M^2$ *for every node $v \in V$,*

(iii) $\beta_v < \log(2M\cdot R)$ *for every node $v \in V\backslash\{\, s \,\}$ with $\sum_{e\in\delta^+_v} x'_e > 0$.*

The proof basically works the following way. We first assume that one component hits the upper boundary, then we define a point in X that equals the solution but only differs in that one component. Together with (VI) we can turn this into a contradiction. For (ii) we use the already proven (NCP) properties for x'_e and the basic observation (2.1) of a static b-transshipment (see page 13). Finally, (iii) is shown by applying (NCP) to x'_e and using flow conservation of $(x'_e)_{e\in E'}$, which is induced by (NCP) applied to ℓ'_v for all $v \in V$. For the sake of clarity we moved the proof to the appendix on page 119.
With this lemma we can finally show the following existence result.

Theorem 6.10.
Consider an acyclic network $G' = (V, E')$ with source s and sink t, outflow capacities $(\nu_e^-)_{e\in E'}$, inflow bounds $(b_e^+)_{e\in E'}$ as well as a set of resetting arcs $E^* \subseteq E'$. Suppose that every node v is reachable by s. Then there exists a spillback thin flow (x', ℓ', c) with resetting on E^*.

To prove this we consider a solution $(x', \ell', \beta) \in \mathrm{SOL}(X, \Gamma)$ and set $c_v := e^{-\beta_v}$. With the help of Lemma 6.9 we can use the (NCP) properties to show the spillback thin flow equations. Note that the nodes with zero throughput of x' need a special treatment. For those nodes we basically set $c_v := 1$ and $\ell_v := \min_{e=uv\in E'} \rho_e(\ell'_u, x'_e, c_v)$. The details of this proof can be found in the appendix on page 120.

6.6 Constructing Nash Flows Over Time

In the same way as in the base model we want to use spillback thin flows in order to construct a Nash flow over time in the kinematic wave model. In contrast to the base model we will have four different kinds of events that can end a phase. In the following we only consider Nash flows over time f that have right-continuous and piece-wise constant in- and outflow rates.

α-Extensions. For a fixed $\phi \in \mathbb{R}_{\geq 0}$ we consider once more a **restricted Nash flow over time** only taking into account the particles in $[0, \phi)$. This is enough to determine the earliest arrival times $\ell_v(\phi)$ for each $v \in V$ and the inflow bounds $b_e^+(\ell_u(\phi))$ for all $e = uv \in E$ as well as E'_ϕ, E^*_ϕ and $\bar{E}_\phi$. Hence, Theorem 6.10 provides a spillback thin flow (x', ℓ', c) on the current shortest paths network $G'_\phi = (V, E'_\phi)$ with resetting on E^*_ϕ and inflow bounds $(b_e^+(\ell_u(\phi))$ for all $e = uv \in E'$. We use this spillback thin flow to extend the restricted Nash flow over time in the following way. We set

$$\ell_v(\phi + \xi) := \ell_v(\phi) + \xi \cdot \ell'_v \qquad \text{and} \qquad x_e(\phi) := x_e(\phi) + \xi \cdot x'_e \quad \text{for } \xi \in [0, \alpha),$$

as well as

$$f_e^+(\theta) := \frac{x'_e}{\ell'_u} \quad \text{for } \theta \in [\ell_u(\phi), \ell_u(\phi+\alpha)) \qquad \text{and} \qquad f_e^-(\theta) = \frac{x'_e}{\ell'_v} \quad \text{for } \theta \in [\ell_v(\phi), \ell_v(\phi+\alpha]).$$

Additionally, we define

$$r(\theta) := c_v \cdot R \quad \text{for all } \theta \in [\ell_s(\phi), \ell_s(\phi+\alpha)).$$

As before this extended flow over time is called **α-extension**.

Thin flow phase. In the following we present some necessary bounds on α, which we later show to be sufficient for the α-extension to form a restricted Nash flow over time on $[0, \phi+\alpha)$.

As in the base model, queues can only shrink until they deplete and non active arcs can become active and open alternative routes. Thus, we get the following two conditions on α for all $e = uv$:

$$\ell_v(\phi) - \ell_u(\phi) + \alpha(\ell'_v - \ell'_u) \geq \tau_e \quad \text{if } e \in E^*_\phi \tag{6.8}$$
$$\ell_v(\phi) - \ell_u(\phi) + \alpha(\ell'_v - \ell'_u) \leq \tau_e \quad \text{if } e \in E \backslash E'_\phi. \tag{6.9}$$

In addition, the inflow bounds of the spillback arcs need to be constant within one phase, i.e., for all $e = uv \in \bar{E}_\phi$ we require that

$$b^+_e(\ell_u(\phi) + \xi \cdot \ell'_u) = b^+_e(\ell_u(\phi)) \quad \text{for all } \xi \in [0, \alpha). \tag{6.10}$$

Finally, the spillback thin flow changes whenever an arc becomes full. Thus, within an extension phase, the total amount of flow on an arc $e = uv \in E'_\phi \backslash \bar{E}_\phi$ stays strictly under the storage capacity:

$$d_e(\ell_u(\phi + \xi)) < \sigma_e \quad \text{for } \xi \in [0, \alpha). \tag{6.11}$$

Note that F^-_e needs not be linear on $[\ell_u(\phi), \ell_u(\phi + \alpha))$. We call $\alpha > 0$ **feasible** if it satisfies Equations (6.8) to (6.11) and the following lemma shows that such an α always exists.

Lemma 6.11. *For a given restricted Nash flow over time on $[0, \phi)$ in the kinematic wave model there exists a feasible $\alpha > 0$.*

Proof. By Lemma 6.6 (ii) and (iii) we have that $\ell_v(\phi) - \ell_u(\phi) > \tau_e$ for $e = uv \in E^*_\phi$ and $\ell_v(\phi) - \ell_u(\phi) < \tau_e$ for $e = uv \in E \backslash E'_\phi$. Since $d_e(\ell_u(\phi)) < \sigma_e$ for $e = uv \in E'_\phi \backslash \bar{E}_\phi$ we can find an $\alpha_1 > 0$ that satisfies Equations (6.8), (6.9) and (6.11). Lemma 6.6 (v) states that $\ell_u(\phi) < \ell_v(\phi)$ for full arcs and since f^-_e is piecewise-constant and right-continuous on $[\ell_u(\phi), \ell_v(\phi))$ so is b^+_e. Hence, there is an $\alpha_2 > 0$ satisfying (6.10). Clearly, $\alpha := \min\{\alpha_1, \alpha_2\} > 0$ is feasible. □

For the maximal feasible α we call the interval $[\phi, \phi + \alpha)$ **thin flow phase**.
These four conditions on α are indeed sufficient, which is shown in the next theorem.

Theorem 6.12.

Consider a restricted Nash flow over time on $[0, \phi)$ and a feasible $\alpha > 0$, then the α-extension is a restricted Nash flow over time on $[0, \phi + \alpha)$. Furthermore, the extended ℓ- and x-functions represent the earliest arrival times and the underlying static flows for all $\varphi \in [0, \phi + \alpha)$.

To prove this we first show that the α-extension is a feasible flow over time, where the fair allocation condition follows from (sTF2) and (sTF3), the inflow condition from (sTF4), and the no-slack condition from (sTF5). Furthermore, the no-deadlock condition follows since the total transit time of each cycle is positive. To show that the extended ℓ-labels correspond to the earliest arrival times we do a quite technical case distinction. With this the Nash flow condition follows immediately. The formal proof can be found in the appendix on page 121.
The next theorem finally shows the existence of Nash flows over time in the kinematic wave model.

Theorem 6.13.

There exists a Nash flow over time in the kinematic wave model.

Proof. The empty flow over time is a restricted Nash flow over time for the empty set $[0, 0)$. For a given restricted Nash flow over time f_i on $[0, \phi_i)$ we choose a maximal feasible $\alpha_i \in (0, \infty]$, which exists due to Lemma 6.11, and extend f_i with Theorem 6.12 to a restricted Nash flow over time f_{i+1} on $[0, \phi_{i+1})$, where $\phi_{i+1} = \phi_i + \alpha_i$. This leads to a strictly increasing sequence $(\phi_i)_{i \in \mathbb{N}}$. Suppose this sequence has a finite limit $\phi_\infty := \lim_{i \to \infty} \phi_i < \infty$. In this case we define a restricted Nash flow over time f^∞ for $[0, \phi_\infty)$ by using the point-wise limits of the x- and ℓ-functions. Note that the functions remain monotonically increasing and bounded (see Theorem 6.5), and therefore, the process can be continued from this limit point. Since this enables us to always extend the Nash flow over time, there cannot be an upper bound on the length of the extension interval because the smallest upper bound would correspond to a limit point, which we can extend again. □

Example. We consider a Nash flow over time in the kinematic wave model depicted in Figure 6.4 on the next page. The first particles $[0, 24)$ take the direct path (s, v, t) and thereby build up queues on vt and on sv as the inflow capacity of vt restricts the outflow of sv to 2. For particle 24, which arrives at v at time 14, the waiting time on vt becomes so long that path (v, w, t) becomes active. As the flow leaving v within this second phase is no longer restricted, the outflow rate of sv increases to 4, causing the queue to decrease. For particle 30 the arc sv becomes full (which happens at time 10), and therefore, we have spillback at the source, which throttles the network inflow rate to 2, as the gap flow arrives at s with a rate of 2 during time $[10, 17)$. For particle 44 two events happen simultaneously: The gap flow of rate 4 finally reaches s at time 17, which means that the network inflow rate can again operate at rate 3, and the queue of arc sv depletes at time 19. After this the Nash flow over time reaches its steady state.

6.7 Further Results

Except for the results presented so far, Nash flows over time in the spillback or kinematic wave model have not been studied much yet. But some results concerning uniqueness and the price of anarchy follow rather easily.

6.7.1 Non-Uniqueness of Earliest Arrival Times

It turns out that the ℓ'-labels in spillback thin flows, and hence the earliest arrival times of a Nash flow over time, are not uniquely determined anymore. As seen in Figure 6.5 on page 114 it can happen in the spillback model (even without kinematic waves) that an arc becomes full at the very same time as a new path becomes active. In the displayed graph the x'_{vt} value can be equal to all values in $[\frac{1}{2}, 2]$, which causes the ℓ'_v-label to lie between $[\frac{1}{2}, 2]$ as well. This also makes sense from a traffic perspective as all traffic users arriving at the highway-exit v know that both routes to their destination are of equal length. As long as the congestion on arc vt does not decrease, they can freely choose between the two paths. Even the extreme case that everyone sticks to the congested highway vt and nobody takes the ring road (v, w, t) would be an equilibrium, as this only increases the congestion on arc sv, which every road user experiences independent of the route choice.
This phenomenon does not only show that there is a continuum of valid spillback thin flows, it also shows that a Nash flow over time can be very chaotic as the spillback thin flow can basically change at every point in time after arc vt became full (as long as the particles with the same spillback thin flow form a measurable set). It also indicates that there are dynamic equilibria with different arrival times, and therefore, one might consider the worst or the best Nash flow over time.

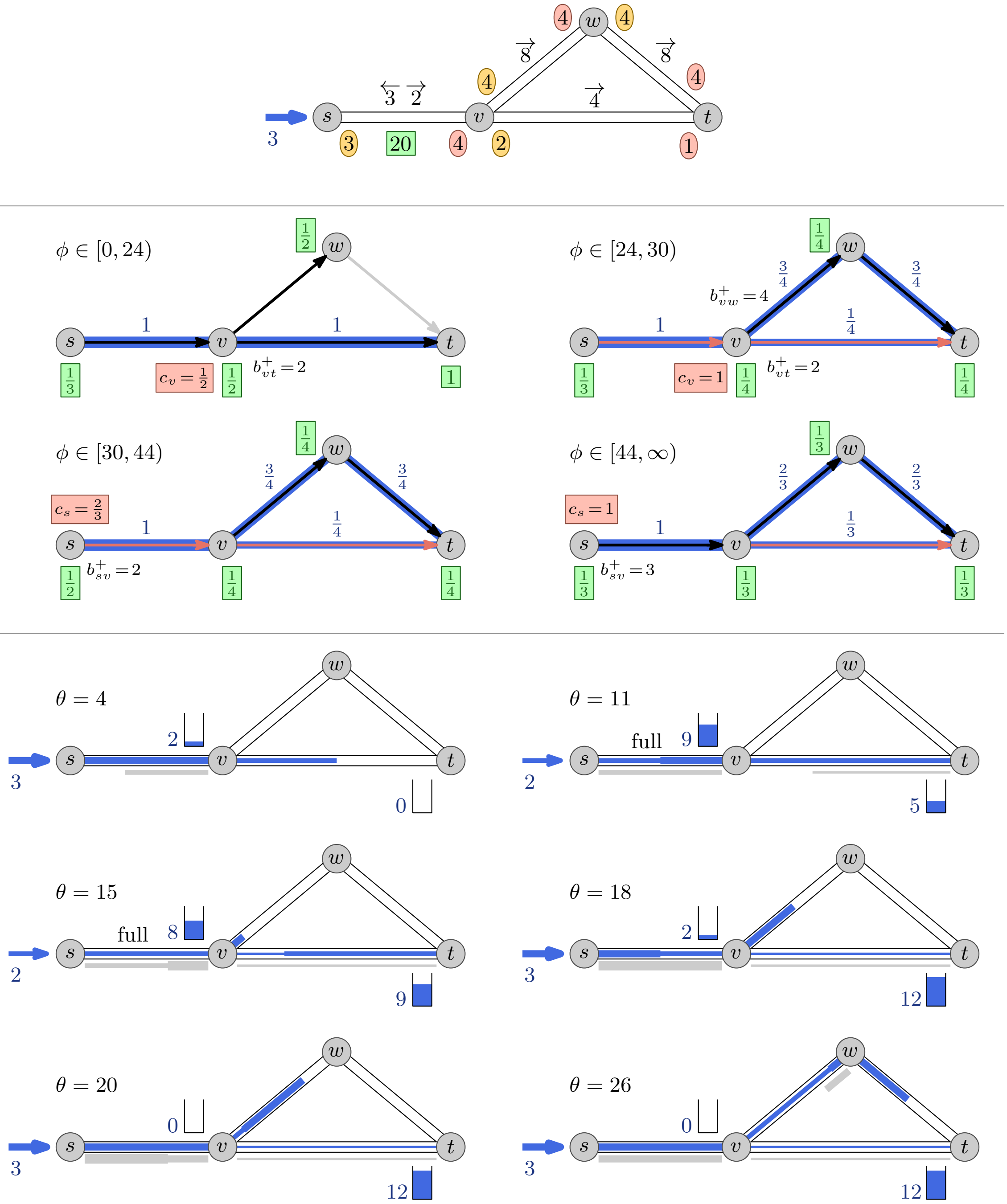


Figure 6.4: An example of a Nash flow over time in the kinematic wave model in the network displayed *on the top*. Here, sv has a transit time $\tau_{sv} = 2$, a gap transit time of $\eta_{sv} = 3$, an outflow capacity of $\nu^-_{sv} = 4$ and a storage capacity of $\sigma_{sv} = 20$. The storage capacities of all other arcs are ∞, and hence, the respective gap transit times are irrelevant. All other arc properties are displayed accordingly and the network inflow rate is 3. The Nash flow over time consists of four thin flow phases depicted *in the middle*. *At the bottom* we show six snapshots in time of the resulting flow over time.

6.7.2 Unbounded Prices of Anarchy

By the network in Figure 6.6 we can observe that the time price of anarchy is unbounded in the spillback model, and therefore also in the kinematic wave model. By setting the inflow capacity ν^{+}_{vt} to a small $\varepsilon > 0$ the Nash flow over time will be unique, as the top path (v, w, t) will never become active. Hence, in a Nash flow over time, the flow will arrive at sink t with a rate of ε, whereas the optimal earliest arrival flow will additionally use the top route. Thus, for a given flow amount A the Nash flow over time needs more than $\frac{A}{\varepsilon}$ time units until the last particle arrives, whereas an optimal flow over time will be done before time $A + 3$. Thus, the time price of anarchy is unbounded for this network family.

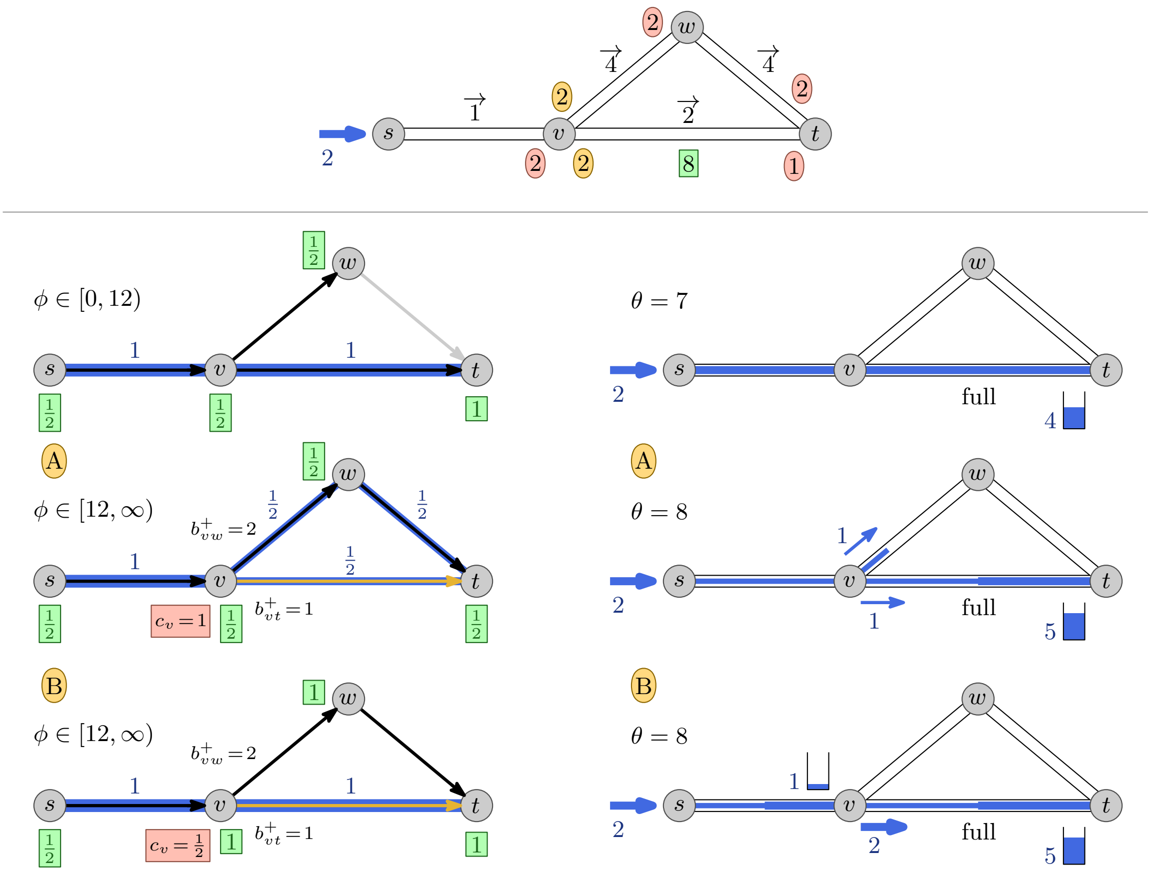


Figure 6.5: This example shows that spillback thin flows are not unique. The network is shown *on the top*. Here, arc vt has a transit time of $\tau_{vt} = 2$, an inflow capacity of $\nu^+_{vt} = 2$, a storage capacity of $\sigma_{vt} = 8$ and an outflow capacity of $\nu^-_{vt} = 1$. The kinematic waves are disabled, i.e., all gap transit times are 0. The Nash flow over time, *displayed below*, consists of two phases. The second phase is not unique, and thus, two possibilities, A and B, are depicted. *On the left* we show the spillback thin flows and *on the right* some key snapshots in time of the resulting flow over time. At the beginning (*top row*) all flow has to take the shortest path (s, v, t) as arc wt is inactive. A queue is growing at the end of arc vt. For particle 12 two events happen at the same time. The arc wt becomes active and, at the moment particle 12 reaches v at time $\theta = 7$, arc vt gets full. Now, there are multiple possibilities for the next spillback thin flow. Either the flow splits up and half of the particles take the top route while the other half continues to use the full arc (*Scenario A*). In this case we have $c_v = 1$ and no queue on arc sv. Or all particles stay on the bottom route (*Scenario B*) in which case we have $c_v = \frac{1}{2}$ and due to spillback a queue is growing on arc sv. All convex combinations of these two spillback thin flows are also valid.

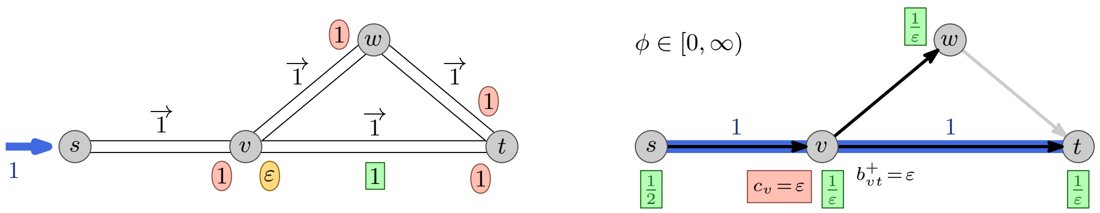


Figure 6.6: This family of networks (*left side*) shows that the time price of anarchy is unbounded in the kinematic wave/spillback model. All transit times and outflow capacities are 1 and arc vt has a storage capacity of $\sigma_{vt} = 1$ and a parameterized inflow capacity of $\nu^+_{vt} = \varepsilon$ for some $\varepsilon \in (0, 1)$. All gap transit times are 0 and all other inflow and storage capacities are ∞ or large enough. As the top route will never be used by any Nash flow over time we only have one thin flow phase and a unique spillback thin flow (*right side*) with $\ell'_t = \frac{1}{\varepsilon}$. Hence, the arrival time increases very fast with slope $\frac{1}{\varepsilon}$.

6.8 Appendix: Technical Proofs

Lemma 6.2. *For a feasible flow over time f the following statements hold for all $\theta \in [0, \infty)$:*

(i) If e is full at time θ we have $z_e(\theta) > 0$.

(ii) If e is full at time θ we have $z_e(\theta - \eta_e) > 0$.

(iii) Every arc that is full at time θ with $f_e^+(\theta) = b_e^+(\theta)$ is part of a (finite) congestion suffix.

(iv) There is a function $\varepsilon\colon [0, \infty) \to (0, 1)$ depending only on the network but not on f such that every arc e with $z_e(\theta) > 0$ satisfies $f_e^-(\theta) \geq \varepsilon(\theta)$ and $b_e^+(\theta) \geq \varepsilon(\theta)$.

Proof.

(i) The inflow condition yields $f_e^+(\theta) \leq \nu_e^+$ and by definition we have $g_e(\theta) \leq \nu_e^-$. Hence,

$$z_e(\theta) = F_e^+(\theta - \tau_e) - F_e^-(\theta) \geq F_e^+(\theta) - \nu_e^+ \cdot \tau_e - F_e^-(\theta) + G_e(\theta) - \nu_e^- \cdot \eta_e \overset{(6.1)}{>} d_e(\theta) - \sigma_e \geq 0.$$

(ii) Again with $f_e^+(\theta) \leq \nu_e^+$ and the definitions of z_e, G_e and d_e we obtain

$$z_e(\theta - \eta_e) = F_e^+(\theta - \eta_e - \tau_e) - F_e^-(\theta - \eta_e) \geq F_e^+(\theta) - \nu_e^+ \cdot (\tau_e + \eta_e) - F_e^-(\theta) + G_e(\theta) \overset{(6.1)}{>} d_e(\theta) - \sigma_e \geq 0.$$

(iii) If e is not throttled at time $\theta - \eta_e$ we are done, since the one-elemental sequence (e) is a congestion suffix. Thus, suppose e is throttled at time $\theta_2 = \theta - \eta_e$. Then, by the no-slack condition, there has to be a consecutive arc e_2 with $f_{e_2}^+(\theta_2) = b_{e_2}^+(\theta_2)$. If e_2 is full at time θ_2 and was throttled at time $\theta_3 := \theta_2 - \eta_{e_2}$ we find an arc e_3 with $f_{e_3}^+(\theta_3) = b_{e_3}^+(\theta_3)$. We repeat this argumentation until we obtain an arc e_k that has $f_{e_k}^+(\theta_k) = b_{e_k}^+(\theta_k)$ but is not full at time θ_k or not throttled at time $\theta_k - \eta_{e_k}$. To show that such an arc e_k exists, assume for contradiction that the constructed sequence $(e_1, e_2, e_3, \dots)$ is unending. Since E is finite, there has to be some node v that is visited infinitely often. But due to the no-deadlock condition each time we visit v we consider a point in time which is strictly earlier (by at least $\min\{\, \eta_e > 0 \mid e \in E \,\}$) than the last visit of v. Since no arc is full at time 0 this is a contradiction.

(iv) We set

$$\nu_{\min} := \min\left(\{\, \nu_e^+, \nu_e^- \mid e \in E \,\} \cup \{\, 1 \,\}\right), \qquad \Sigma := \max\left\{ \sum_{e \in E} \nu_e^-, 1 \right\},$$

$$\eta_{\min} := \min\left(\{\, \eta_e > 0 \mid e \in E \,\} \cup \{\, 1 \,\}\right) \quad \text{and} \quad \varepsilon(\theta) := \left(\frac{\nu_{\min}}{\Sigma}\right)^{|E| \cdot \frac{\theta}{\eta_{\min}}} \cdot \nu_{\min}.$$

If e is not throttled at time θ we have $f_e^-(\theta) = \nu_e^-$. Thus, suppose e is throttled. By the no-slack condition there has to be a consecutive arc e_1 with $f_{e_1}^+(\theta) = b_{e_1}^+(\theta)$. Due to (iii) there has to be a congestion suffix $(e_1, e_2, \dots, e_k)$ at time $\theta_1 = \theta$ where $k \leq \frac{\theta}{\eta_{\min}} \cdot |E|$. If e_k is not full we have $f_{e_k}^+(\theta_k) = b_{e_k}^+(\theta_k) = \nu_{e_k}^+$. If e_k is full but not throttled at time $\theta_{k+1} := \theta_k - \eta_{e_k}$ we obtain by (ii) that e_k had a queue at time θ_{k+1}, and therefore $g_{e_k}(\theta_k) = \nu_{e_k}^-$, which leads to $f_{e_k}^+(\theta_k) = b_{e_k}^+(\theta_k) = \min\{\, \nu_{e_k}^+, \nu_{e_k}^- \,\}$. Furthermore, for two consecutive arcs $e_i = uv$ and $e_{i+1} = vw$ we have that

$$f_{e_i}^-(\theta_{i+1}) = c_v(\theta_{i+1}) \cdot \nu_{e_i}^- \geq \frac{\sum_{e' \in \delta^+(v)} f_{e'}^+(\theta_{i+1})}{\sum_{e' \in \delta^-(v)} \nu_{e'}^-} \cdot \nu_{\min} \geq \frac{f_{e_{i+1}}^+(\theta_{i+1})}{\Sigma} \cdot \nu_{\min}. \tag{6.12}$$

Since the arc e_i, for $i = 1, \dots, k-1$, is full at time θ_i with exhausted inflow capacity it holds that $f^+_{e_i}(\theta_i) = b^+_{e_i}(\theta_i) = \min\{\, f^-_{e_i}(\theta_{i+1}), \nu^-_{e_i} \,\}$. Recursive application of (6.12) along the sequence gives

$$f^-_e(\theta) \geq \left(\frac{\nu_{\min}}{\Sigma}\right)^k \cdot \nu_{\min} \geq \varepsilon(\theta).$$

With (ii) we obtain that $b^+_e(\theta) = \min\{\, f^-_e(\theta - \eta_e), \nu^+_e \,\} \geq \varepsilon(\theta - \eta_e) \geq \varepsilon(\theta)$ in the case that e is full. □

Lemma 6.3. *For a feasible flow over time f it holds for all $e \in E$, $v \in V$ and $\theta \in [0, \infty)$ that:*

(i) $q_e(\theta) > 0 \quad \Leftrightarrow \quad z_e(\theta + \tau_e) > 0.$

(ii) $z_e(\theta + \tau_e + \xi) > 0$ *for all* $\xi \in [0, q_e(\theta))$.

(iii) $F^+_e(\theta) = F^-_e(T_e(\theta))$ *whenever* $T_e(\theta) < \infty$.

(iv) For $\theta_1 < \theta_2$ *with* $F^+_e(\theta_2) - F^+_e(\theta_1) = 0$ *and* $z_e(\theta_2 + \tau_e) > 0$ *we have* $T_e(\theta_1) = T_e(\theta_2)$.

(v) If $T_e(\theta) < \infty$ *and* $f^-_e(T_e(\theta)) = 0$ *then* $F^+_e(\theta + q_e(\theta)) - F^+_e(\theta) = 0$.

(vi) If $T_e(\theta) < \infty$ *the push rate functions satisfy*

$$b^-_e(T_e(\theta)) = \begin{cases} \nu^-_e & \textit{if } F^+_e(\theta + q_e(\theta)) - F^+_e(\theta) > 0, \\ \min\{\, f^+_e(T_e(\theta) - \tau_e), \nu^-_e \,\} & \textit{else.} \end{cases}$$

(vii) The functions T_e *are monotonically increasing.*

(viii) The functions T_e *and* q_e *are differentiable at almost all* θ *with* $q_e(\theta) < \infty$.

(ix) For almost all $\theta \in [0, \infty)$ *with* $q_e(\theta) < \infty$ *we have*

$$q'_e(\theta) = \begin{cases} \frac{f^+_e(\theta)}{f^-_e(T_e(\theta))} - 1 & \textit{if } f^-_e(T_e(\theta)) > 0, \\ -1 & \textit{else if } z_e(\theta + \tau_e) > 0, \\ 0 & \textit{else.} \end{cases}$$

Proof.

(i) This follows directly by the definition of q_e.

(ii) For $q_e(\theta) < \infty$ we have by definition that $q_e(\theta)$ is the minimal value such that

$$F^-_e(\theta + \tau_e + q_e(\theta)) - F^-_e(\theta + \tau_e) = z_e(\theta + \tau_e) = F^+_e(\theta) - F^-_e(\theta + \tau_e) \tag{6.13}$$

and for $q_e(\theta) = \infty$ we have for all $q \in [0, \infty)$ that

$$F^-_e(\theta + \tau_e + q) - F^-_e(\theta + \tau_e) < z_e(\theta + \tau_e) = F^+_e(\theta) - F^-_e(\theta + \tau_e).$$

In both cases we obtain $F^+_e(\theta) - F^-_e(\theta + \tau_e + \xi) > 0$ for $\xi \in [0, q_e(\theta))$. Since F^+_e is monotonically increasing we have for all $\xi \in [0, q_e(\theta))$ that

$$z_e(\theta + \tau_e + \xi) = F^+_e(\theta + \xi) - F^-_e(\theta + \tau_e + \xi) \geq F^+_e(\theta) - F^-_e(\theta + \tau_e + \xi) > 0.$$

(iii) Whenever $T_e(\theta) < \infty$ we have $q_e(\theta) < \infty$, and therefore, (6.13) implies that $F_e^-(T_e(\theta)) = F_e^+(\theta)$ in this case.

(iv) We have

$$\begin{aligned}
q_e(\theta_1) &= \min\left\{ q \geq 0 \;\middle|\; \int_{\theta_1+\tau_e}^{\theta_2+\tau_e} f_e^-(\xi)\,\mathrm{d}\xi + \int_{\theta_2+\tau_e}^{\theta_1+\tau_e+q} f_e^-(\xi)\,\mathrm{d}\xi = F_e^+(\theta_1) - F_e^-(\theta_1+\tau_e) \right\} \\
&= \min\left\{ p = q - \theta_2 + \theta_1 \geq 0 \;\middle|\; \int_{\theta_2+\tau_e}^{\theta_2+\tau_e+p} f_e^-(\xi)\,\mathrm{d}\xi = F_e^+(\theta_2) - F_e^-(\theta_2+\tau_e) \right\} + \theta_2 - \theta_1 \\
&= q_e(\theta_2) + \theta_2 - \theta_1.
\end{aligned}$$

Thus, $T_e(\theta_1) = \theta_1 + \tau_e + q_e(\theta_1) = \theta_2 + \tau_e + q_e(\theta_2) = T_e(\theta_2)$. Note that $q_e(\theta_1) = \infty$ if, and only if, the set above is empty, which is exactly the case when $q_e(\theta_2) = \infty$. Hence, $T_e(\theta_1) = \infty = T_e(\theta_2)$ in this case.

(v) In order to show the contra-position assume that $F_e^+(\theta + q_e(\theta)) - F_e^+(\theta) > 0$. We obtain

$$z_e(T_e(\theta)) = F_e^+(\theta + q_e(\theta)) - F_e^-(T_e(\theta)) \overset{\text{(iii)}}{=} F_e^+(\theta + q_e(\theta)) - F_e^+(\theta) > 0.$$

Thus, Lemma 6.2 (iv) implies $f_e^-(T_e(\theta)) > \varepsilon(\theta) > 0$.

(vi) This follows by $z_e(T_e(\theta)) \overset{\text{(iii)}}{=} F_e^+(\theta + q_e(\theta)) - F_e^+(\theta)$ and the definition of $b_e^-(\theta)$.

(vii) Consider two points in time $\theta_1 < \theta_2$. We show that $T_e(\theta_1) \leq T_e(\theta_2)$.

If $T_e(\theta_1) = \infty$ we have $q_e(\theta_1) = \infty$, which means

$$\int_{\theta_1+\tau_e}^{\infty} f_e^-(\xi)\,\mathrm{d}\xi < z_e(\theta_1+\tau_e) = F_e^+(\theta_1) - F_e^-(\theta_1+\tau_e).$$

Hence, it holds that

$$\int_{\theta_2+\tau_e}^{\infty} f_e^-(\xi)\,\mathrm{d}\xi = \int_{\theta_1+\tau_e}^{\infty} f_e^-(\xi)\,\mathrm{d}\xi - \int_{\theta_1+\tau_e}^{\theta_2+\tau_e} f_e^-(\xi)\,\mathrm{d}\xi < F_e^+(\theta_1) - F_e^-(\theta_2+\tau_e) \leq z_e(\theta_2+\tau_e)$$

implying $q_e(\theta_2) = \infty$, and therefore, $T_e(\theta_2) = \infty$.

Now suppose that $T_e(\theta_1) < \infty$. If $T_e(\theta_2) = \infty$ we already have $T_e(\theta_1) \leq T_e(\theta_2)$, thus, we consider the case that $T_e(\theta_2) < \infty$. Since F_e^+ is monotonically increasing, (iii) implies that

$$F_e^-(T_e(\theta_2)) = F_e^+(\theta_2) \geq F_e^+(\theta_1) = F_e^-(T_e(\theta_1)). \tag{6.14}$$

If this holds with strict inequality, we obtain by monotonicity of F_e^- that $T_e(\theta_1) < T_e(\theta_2)$. If (6.14) holds with equality we have two cases. If $z_e(\theta_2+\tau_e) > 0$, (iv) states that $T_e(\theta_1) = T_e(\theta_2)$. If $z_e(\theta_2+\tau_e) = 0$ the statement in (ii) applied to θ_1 yields $\xi := \theta_2 - \theta_1 \notin [0, q_e(\theta_1))$. Thus,

$$T_e(\theta_2) \overset{\text{(i)}}{=} \theta_2 + \tau_e \geq \theta_1 + \tau_e + q_e(\theta_1) = T_e(\theta_1).$$

(viii) Note that $T_e(\theta) < \infty$ if, and only if, $q_e(\theta) < \infty$. By (vii) T_e is monotonically increasing on the set $\{\, \theta \in [0,\infty) \mid T_e(\theta) < \infty \,\}$, and hence, Lebesgue's theorem for the differentiability of monotone functions (Theorem 2.5 on page 19) states that T_e is differentiable at almost all points in time in this set. As a sum of almost everywhere differentiable functions $q_e(\theta) = T_e(\theta) - \tau_e - \theta$ is also almost everywhere differentiable on this set.

(ix) By (iii) we have $F_e^-(\theta + \tau_e + q_e(\theta)) = F_e^+(\theta)$. Since the functions F_e^-, F_e^+ and q_e are almost everywhere differentiable we can take the derivative on both sides to obtain with the chain rule that

$$f_e^-(T_e(\theta)) \cdot (1 + q_e'(\theta)) = f_e^+(\theta).$$

Hence, $q_e'(\theta) = \frac{f_e^+(\theta)}{f_e^-(T_e(\theta))} - 1$ if $f_e^-(T_e(\theta)) > 0$. In the case of $f_e^-(T_e(\theta)) = 0$ and $z_e(\theta + \tau_e) > 0$ (v) yields $F_e^+(\theta + \xi) - F_e^+(\theta) = 0$ for all $\xi \in [0, q_e(\theta)) \neq \emptyset$, and therefore $T_e(\theta) = T_e(\theta + \xi)$ by (i) and (iv). It follows that

$$q_e(\theta + \xi) = T_e(\theta + \xi) - \theta - \xi - \tau_e = T_e(\theta) - \theta - \tau_e - \xi = q_e(\theta) - \xi.$$

Thus, the right derivative of q_e at θ equals -1, which implies that either q is not differentiable at θ or $q_e'(\theta) = -1$.

Finally, in the case of $f_e^-(T_e(\theta)) = 0$ and $z_e(\theta + \tau_e) = 0$ we have $q_e(\theta) = 0$ by (i), and thus, θ is a local minimum of q_e. Hence, q_e is either not differentiable at θ or $q_e'(\theta) = 0$. □

Lemma 6.6. *Given a Nash flow over time the following holds for all times θ:*

(i) $E_\phi^* \subseteq E_\phi'$.

(ii) $E_\phi' = \{\, e = uv \mid \ell_v(\phi) \geq \ell_u(\phi) + \tau_e \,\}$.

(iii) $E_\phi^* = \{\, e = uv \mid \ell_v(\phi) > \ell_u(\phi) + \tau_e \,\}$.

(iv) $\bar{E}_\phi \subseteq E_\phi'$.

(v) $\ell_u(\phi) < \ell_v(\phi)$ *for all* $e = uv \in \bar{E}_\phi$.

Proof. (i) to (iii) follow from the statements made in Lemma 3.4 of the base model. The proof of this lemma can be found on page 49. The last two statements are shown in the following.

(iv) For $e \in \bar{E}_\phi$ Lemma 6.2 (ii) states that $z_e(\ell_u(\phi) - \eta_e) > 0$. Therefore, by continuity of z_e and Lemma 6.2 (iv) we have that $f_e^-(\xi) > 0$ for all $\xi \in [\ell_u(\phi) - \eta_e - \delta, \ell_u(\phi) - \eta_e]$ for some small $\delta > 0$. Considering the amount of flow that has left the arc and whose gap flow has also already left the arc, we obtain for all $\varepsilon > 0$ that

$$\begin{aligned} F_e^-(\ell_u(\phi)) - G_e(\ell_u(\phi)) &= \int_0^{\ell_u(\phi) - \eta_e} f_e^-(\xi)\,\mathrm{d}\xi \\ &> \int_0^{\ell_u(\phi) - \eta_e - \varepsilon} f_e^-(\xi)\,\mathrm{d}\xi \\ &= F_e^-(\ell_u(\phi) - \varepsilon) - G_e(\ell_u(\phi) - \varepsilon). \end{aligned}$$

This, together with $d_e(\ell_u(\phi)) = \sigma_e \geq d_e(\ell_u(\phi) - \varepsilon)$, yields

$$\begin{aligned} F_e^+(\ell_u(\phi)) &= d_e(\ell_u(\phi)) + F_e^-(\ell_u(\phi)) - G_e(\ell_u(\phi)) \\ &> d_e(\ell_u(\phi) - \varepsilon) + F_e^-(\ell_u(\phi) - \varepsilon) - G_e(\ell_u(\phi) - \varepsilon) \\ &= F_e^+(\ell_u(\phi) - \varepsilon). \end{aligned}$$

Hence, Lemma 3.3 (v) implies $e \in E_\phi'$.

(v) Due to (iv), e is active, i.e., $\ell_u(\phi) + \tau_e + q_e(\ell_u(\phi)) = \ell_v(\phi)$. Thus, for $\tau_e > 0$ the claim follows immediately. For $\tau_e = 0$ we get by Lemma 6.2 (i) that $0 < z_e(\ell_u(\phi)) = z_e(\ell_u(\phi) + \tau_e)$. Hence, Lemma 6.3 (i) implies $q_e(\ell_u(\phi)) > 0$, and therefore $\ell_u(\phi) < \ell_v(\phi)$. □

Lemma 6.9. *For every solution $(x', \ell', \beta) \in \mathrm{SOL}(X, \Gamma)$ we have*

(i) $x'_e < 4M^2 \cdot \nu_e^-$ *for every arc* e *,*

(ii) $\ell'_v < 3M^2$ *for every node* $v \in V$*,*

(iii) $\beta_v < \log(2M \cdot R)$ *for every node* $v \in V \setminus \{ s \}$ *with* $\sum_{e \in \delta_v^+} x'_e > 0$.

Proof.

(i) Suppose there is an arc $e \in E'$ with $x'_e = 4M^2 \cdot \nu_e^-$. Note that $e^{-\beta_v} \leq 1$ and $\ell'_v \leq 3M^2$, and therefore, $\Gamma_e(x', \ell', \beta) = \frac{x'_e}{e^{-\beta_v} \cdot \nu_e^-} - \ell'_v$ even if $e \in E' \setminus E^*$. Hence, for $(y, \ell', \beta) \in X$ with $y_e := 0$ and $y_i := x'_i$ for $i \in E' \setminus \{ e \}$, (VI) states that

$$0 \leq -x'_e \cdot \left(\frac{x'_e}{e^{-\beta_v} \cdot \nu_e^-} - \ell'_v \right) \leq 4M^2 \cdot \nu_e^- \cdot (\ell'_v - 4M^2).$$

But this is a contradiction since $\ell'_v - 4M^2 < 0$.

(ii) Using (x', k, β) with $k_v = \ell'_v$ for $v \neq s$ we obtain with (VI) that $(k_s - \ell'_s) \cdot (\ell'_s - \frac{1}{e^{-\beta_s} \cdot R}) \geq 0$ for all $k_s \in [0, 3M^2]$. Hence, $\ell'_s = \frac{1}{e^{-\beta_s} \cdot R} \leq 2M < 3M^2$.

Furthermore, we will show that

$$\sum_{e \in \delta_v^-} x'_e \leq \sum_{e \in \delta_v^+} x'_e \qquad \text{for all } v \in V \setminus \{ s, t \}.$$

If $\ell'_v > 0$ this follows from (VI) for $(x', k, \beta) \in X$ with $k_u = \ell'_u$ for all nodes $u \in V \setminus \{ v \}$ and $k_v = 0$. In the case of $\ell'_v = 0$ the inequality follows since (i) and (NCP) imply that $x'_e = 0$ on all arcs $e \in \delta_v^-$.

If we define

$$b(v) := \sum_{e \in \delta_v^+} x'_e - \sum_{e \in \delta_v^-} x'_e \qquad \text{for all } v \in V,$$

the flow x'_e is a feasible static b-transshipment, where $b(v) \geq 0$ for all $v \in V \setminus \{ t \}$. Note that s has no incoming arcs, and thus, $b(s) = \sum_{e \in \delta_s^+} x'_e \geq 0$. Since the graph G' is acyclic and t is the only sink in this b-transshipment, we get that $\sum_{e \in \delta_t^+} x'_e = 0$, and therefore, the definition of Γ_t and (VI) imply $b(t) \geq -1$. In the following we show that a label of $3M^2$ would induce a flow of $x'_e > 1$ on an arc, which is a contradiction to Equation (2.1) on page 13. Suppose there is a node w with $\ell'_w = 3M^2$. Since $\ell'_s < 3M^2$, there has to be an arc $e = uv$ along an s-w-path, such that $\ell'_u < \ell'_v = 3M^2$. By (i) we can apply (NCP) on Γ_e to obtain

$$x'_e \geq \ell'_v \cdot e^{-\beta_v} \cdot \nu_e^- \geq 3M^2 \cdot e^{-\log(2M \cdot R)} \cdot \nu_{\min}^- > M \cdot \frac{1}{R} \cdot \nu_{\min}^- \overset{(6.7)}{\geq} 1.$$

Thus, $\ell'_v < 3M^2$ for every $v \in V$ and by (NCP) flow conservation follows.

$$\sum_{e \in \delta_v^+} x'_e - \sum_{e \in \delta_v^-} x'_e = \begin{cases} 1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{else.} \end{cases} \tag{6.15}$$

In other words, x'_e is a static s-t flow of value 1. Hence, $x'_e \leq 1$ for all $e \in E'$.

(iii) Suppose $\beta_v = \log(2M \cdot R)$ for some $v \in V$ with $\sum_{e\in\delta_v^+} x'_e > 0$. For $(x', \ell', \gamma) \in X$ with $\gamma_u := \beta_u$ for all $u \neq v$ and $\gamma_v := 0$ we obtain from (VI) that $\ell'_v \leq \max_{e\in\delta_v^+} \frac{x'_e}{b_e}$. Let $e_1 \in \arg\max_{e\in\delta_v^+} \frac{x'_e}{b_e}$. For $v = s$ we have a contradiction, since

$$\ell'_s = \frac{1}{e^{-\beta_s} \cdot R} = 2M \overset{(6.7)}{>} \frac{1}{b^+_{\min}} \geq \frac{x'_{e_1}}{b^+_{e_1}}.$$

For $v \neq s$ Equation (6.15) implies that there is at least one incoming arc $e_2 = uv$ that carries $x'_{e_2} \geq \frac{x'_{e_1}}{|\delta_v^-|} \geq \frac{x'_{e_1}}{|E'|} > 0$ flow. Using (NCP) for arc e_2 yields $\Gamma_{e_2}(x', \ell', \beta) = 0$, and therefore, we obtain the following

$$\ell'_v \geq \frac{x'_{e_2}}{e^{-\beta_v} \cdot \nu^-_{e_2}} \geq \frac{x'_{e_1} \cdot e^{\log(2M\cdot R)}}{|E'| \cdot \nu^-_{e_2}} \overset{(6.7)}{\geq} \frac{x'_{e_1} \cdot 2 \cdot \nu^-_{\max} \cdot |E'| \cdot R}{|E'| \cdot \nu^-_{e_2} \cdot R \cdot b^+_{\min}} > \frac{x'_{e_1}}{b^+_{\min}} \geq \frac{x'_{e_1}}{b^+_e}.$$

This is again a contradiction, which finishes the proof. □

Theorem 6.10.

Consider an acyclic network $G' = (V, E')$ with source s and sink t, outflow capacities $(\nu^-_e)_{e\in E'}$, inflow bounds $(b^+_e)_{e\in E'}$ as well as a set of resetting arcs $E^* \subseteq E'$. Suppose that every node v is reachable by s. Then there exists a spillback thin flow (x', ℓ', c) with resetting on E^*.

Proof. Let $(x', \tilde{\ell}', \beta)$ be a solution ofo $\mathrm{VI}(X, \Gamma)$. In order to obtain a spillback thin flow we need to make some modifications. Let $V_0 \subseteq V \setminus \{s\}$ be the set of nodes with

$$\sum_{e\in\delta_v^-} x'_e = \sum_{e\in\delta_v^+} x'_e = 0.$$

We set $c_v = 1$ if $v \in V_0$ and $c_v = e^{-\beta_v}$ otherwise. Note that we have $\rho_e(\cdot, x'_e, e^{-\beta_v}) = \rho_e(\cdot, x'_e, c_v)$ because $c_v \neq e^{-\beta_v}$ implies $x'_e = 0$. Furthermore, let

$$L := \left\{ k \in \mathbb{R}^V_{\geq 0} \;\middle|\; k_v = \tilde{\ell}_v \text{ for } v \in V\setminus V_0 \text{ and } k_v \leq \min_{e=uv\in E'} \rho_e(k_u, x'_e, c_v) \text{ for } v \in V \right\}.$$

Clearly, $\tilde{\ell}' \in L$ since for every $v \in V$ we obtain by (NCP) applied to $e = uv$ that

$$\tilde{\ell}'_v \leq \left\{ \begin{array}{ll} \frac{x'_e}{e^{-\beta_v}\cdot\nu^-_e} & \text{if } e \in E^* \\ \max\left\{\tilde{\ell}'_u, \frac{x'_e}{e^{-\beta_v}\cdot\nu^-_e}\right\} & \text{if } e \in E'\setminus E^* \end{array} \right\} = \rho_e(\tilde{\ell}'_u, x'_e, e^{-\beta_v}) = \rho_e(\tilde{\ell}'_u, x'_e, c_v).$$

So L is non-empty and closed. From the facts that x'_e and $\tilde{\ell}'_s = \frac{1}{e^{-\beta_s}} \leq 2M$ are bounded and that every node is reachable from s, it follows that L is also bounded, i.e., we can define $\ell' := \arg\max_{k\in L} \sum_{v\in V} k_v$.

In the remaining part of the proof we show that (x', ℓ', c) is a spillback thin flow. Equation (6.15) states that x' is a static s-t-flow of value 1, so it remains to show Equations (sTF1) to (sTF5).

(sTF1) Applying (NCP) to ℓ'_s yields $(\ell'_s - \frac{1}{e^{-\beta_s}}) \geq 0$ and $\ell'_s \cdot (\ell'_s - \frac{1}{e^{-\beta_s}}) = 0$. Thus, $\ell'_s = \frac{1}{e^{-\beta_s}} = \frac{1}{c_s}$.

(sTF3) Applying (NCP) to x'_e yields

$$x'_e \cdot \left(\rho_e(\tilde{\ell}'_u, x'_e, e^{-\beta_v}) - \tilde{\ell}'_v\right) = 0.$$

So if $x'_e > 0$ it follows that $u, v \notin V_0$, and therefore $\tilde{\ell}'_v = \ell'_v$ and $\tilde{\ell}'_u = \ell'_u$, which shows

$$\ell'_v = \rho_e(\ell'_u, x'_e, e^{-\beta_v}) = \rho_e(\ell'_u, x'_e, c_v).$$

(sTF2) The definition of L implies

$$\ell'_v \leq \min_{e=uv\in E'} \rho_e(\ell'_u, x'_e, c_v).$$

In order to show equality we consider two cases. If $v \notin V_0$ there exists an $e \in \delta_v^-$ with $x'_e > 0$, and hence, (sTF3) implies equality.
For $v \in V_0$ suppose for contradiction that $\ell'_v < \min_{e=uv\in E'} \rho_e(\ell'_u, x'_e, c_v)$. We set $k_w := \ell'_w$ for $w \in V \setminus \{ v \}$ and $k_v = \min_{e=uv\in E'} \rho_e(\ell'_u, x'_e, c_v)$. Since $\rho_e(\cdot, x'_e, c_w)$ is non-decreasing we have

$$k_w = \ell'_w \leq \rho_e(\ell'_u, x'_e, c_w) \leq \rho_e(k_u, x'_e, c_w)$$

for $w \neq v$. Hence, we obtain that $k \in L$ which is a contradiction to the maximality of ℓ'_v.

(sTF4) Applying (NCP) to $\bar{v} \in \bar{V}$ yields $\tilde{\ell}'_v - \max \frac{x'_e}{b_e^+} \geq 0$, which proves (sTF4) for $v \notin V_0$. For $v \in V_0$ we have $\ell'_v \geq 0 = \max_{e=vw\in E'} \frac{x'_e}{b_e^+}$, trivially.

(sTF5) Finally, we have $\beta_v \cdot \left(\tilde{\ell}_v - \max_{e=vw\in E'} \frac{x'_e}{b_e^+}\right) = 0$, which implies (sTF5) for $v \notin V_0$, since $\beta_v > 0$ means that $c_v = e^{-\beta_v} < 1$, and thus, we have equality in this case. For $v \in V_0$ we set $c_v = 1$, and therefore there is nothing to show.
Hence, (x', ℓ', c) forms a spillback thin flow, which finishes the proof of Theorem 6.10. ☐

Theorem 6.12.
Consider a restricted Nash flow over time on $[0, \phi)$ and a feasible $\alpha > 0$, then the α-extension is a restricted Nash flow over time on $[0, \phi + \alpha)$. Furthermore, the extended ℓ- and x-functions represent the earliest arrival times and the underlying static flows for all $\varphi \in [0, \phi + \alpha)$.

Proof. Obviously f_e^- and f_e^+ are bounded, piece-wise constant and right-continuous. All conditions are fulfilled on $[0, \phi)$ since nothing has changed on this interval. Note that we use the linearly extended ℓ-labels in the first part of the proof and that we show only in the end that they are indeed the earliest arrival times.

Flow conservation. For $\ell'_v > 0$ we obtain for all $v \in V \setminus \{ t \}$ and all $\theta \in [\ell_v(\phi), \ell_v(\phi + \alpha))$ that

$$\sum_{e\in\delta^+(v)} f_e^+(\theta) - \sum_{e\in\delta^-(v)} f_e^-(\theta) = \sum_{e\in\delta^+(v)} \frac{x'_e}{\ell'_v} - \sum_{e\in\delta^-(v)} \frac{x'_e}{\ell'_v} = \begin{cases} 0 & \text{if } v \in V \setminus \{ s, t \}, \\ c_s \cdot R = r(\theta) \in (0, R] & \text{if } v = s. \end{cases}$$

Note that x' is a static flow of value 1 and that $\ell'_s = \frac{1}{c_s \cdot R}$. For the degenerated case of $\ell'_v = 0$ we have $[\ell_v(\phi), \ell_v(\phi + \alpha)) = \emptyset$, and therefore, there is nothing to show.

x is well-defined. For all $\xi \in [0, \alpha)$ we have

$$\begin{aligned} F_e^+(\ell_u(\phi + \xi)) &= x_e(\phi) + \int_{\ell_u(\phi)}^{\ell_u(\phi)+\xi\cdot\ell'_u} f_e^+(\theta)\, \mathrm{d}\theta = x_e(\phi) + \xi \cdot x'_e = x_e(\phi + \xi) \quad \text{and} \\ F_e^-(\ell_v(\phi + \xi)) &= x_e(\phi) + \int_{\ell_v(\phi)}^{\ell_v(\phi)+\xi\cdot\ell'_v} f_e^-(\theta)\, \mathrm{d}\theta = x_e(\phi) + \xi \cdot x'_e = x_e(\phi + \xi). \end{aligned} \tag{6.16}$$

Fair allocation condition. For every arc $e = uv$ we have to show that

$$f_e^-(\ell_v(\phi+\xi)) = \min\left\{\, b_e^-(\ell_v(\phi+\xi)), c_v \cdot \nu_e^- \,\right\} \qquad \text{for } \xi \in [0,\alpha).$$

This is obvious for $\ell'_v = 0$, so we assume $\ell'_v > 0$.

Case 1: $x'_e = 0$.
Either e is not active or it is active, but then $\ell'_v > 0$ together with (sTF2) implies that e is not resetting. Either way $z_e(\ell_u(\phi) + \tau_e) = 0$ and since $f_e^+(\ell_u(\phi+\xi)) = 0$ the queue stays empty. We have

$$f_e^-(\ell_v(\phi+\xi)) = \frac{x'_e}{\ell'_v} = 0 \qquad \text{and} \qquad b_e^-(\ell_v(\phi+\xi)) = f_e^+(\ell_v(\phi+\xi) - \tau_e) = 0$$

for $\xi \in [0,\alpha)$ since either $\ell_v(\phi+\xi) - \tau_e \geq \ell_u(\phi)$ or $\ell_v(\phi+\xi) - \tau_e < \ell_u(\phi)$. In the first case the inflow is part of either the current or an even later spillback thin flow. Either way the inflow is 0. In the second case the inflow if part of a previous spillback thin flow, thus, e is not active for ζ with $\ell_u(\zeta) = \ell_v(\phi+\xi) - \tau_e$, since

$$T_e(\zeta) = \ell_u(\zeta) + \tau_e + q_e(\zeta) \geq \ell_v(\phi+\xi) > \ell_v(\zeta).$$

We constructed the flow over time on $[0,\phi)$ in such a way that the Nash flow condition is fulfilled at every point in time, and therefore, we have $f_e^+(\ell_v(\phi+\xi) - \tau_e) = f_e^+(\ell_u(\zeta)) = 0$ for all $\xi \in [0,\alpha)$.

Case 2: $x'_e > 0$ and $e \in E'_\phi \backslash E^*_\phi$ with $\frac{x'_e}{c_v \cdot \nu_e^-} \leq \ell'_u$.
It follows from (sTF3) that $\ell'_v = \ell'_u$, and thus,

$$f_e^+(\ell_u(\phi+\xi)) = \frac{x'_e}{\ell'_u} = \frac{x'_e}{\ell'_v} = f_e^-(\ell_v(\phi+\xi)) \qquad \text{for } \xi \in [0,\alpha).$$

We obtain

$$f_e^+(\ell_v(\phi+\xi) - \tau_e) = f_e^+(\ell_v(\phi) - \tau_e + \ell'_v \cdot \xi)) = f_e^+(\ell_u(\phi) + \ell'_u \cdot \xi)) = f_e^+(\ell_u(\phi+\xi)) = f_e^-(\ell_v(\phi+\xi)).$$

This equality yields

$$z_e(\ell_v(\phi+\xi)) = z_e(\ell_v(\phi)) + \int_{\ell_v(\phi)}^{\ell_v(\phi+\xi)} f_e^+(\zeta - \tau_e) - f_e^-(\zeta)\,\mathrm{d}\zeta = 0.$$

By the case distinction we have

$$b_e^-(\ell_v(\phi+\xi)) = f_e^+(\ell_v(\phi+\xi) - \tau_e) = \frac{x'_e}{\ell'_u} \leq c_v \cdot \nu_e^-.$$

In conclusion, we obtain

$$\min\left\{\, b_e^-(\ell_v(\phi+\xi)), c_v \cdot \nu_e^- \,\right\} = b_e^-(\ell_v(\phi+\xi)) = f_e^+(\ell_v(\phi+\xi) - \tau_e) = f_e^-(\ell_v(\phi+\xi)).$$

Case 3: $x'_e > 0$ and ($e \in E^*_\phi$ or $e \in E'_\phi \backslash E^*_\phi$ with $\frac{x'_e}{c_v \cdot \nu_e^-} > \ell'_u$).
It follows from (sTF3) that $\ell'_v = \frac{x'_e}{c_v \cdot \nu_e^-}$, and thus,

$$f_e^-(\ell_v(\phi+\xi)) = \frac{x'_e}{\ell'_v} = c_v \cdot \nu_e^- \qquad \text{for } \xi \in [0,\alpha).$$

It remains to show that $b_e^-(\ell_v(\phi+\xi)) \geq c_v \cdot \nu_e^-$. For $e \in E_\phi^*$ we obtain by (6.8) that

$$\ell_v(\phi) - \ell_u(\phi) + \xi \cdot (\ell_v' - \ell_u') > \tau_e \quad \text{for } \xi \in [0, \alpha).$$

For $e \in E_\phi' \backslash E_\phi^*$ and $\ell_v' = \frac{x_e'}{c_v \cdot \nu_e^-} > \ell_u'$ it follows that $\ell_v(\phi) - \ell_u(\phi) = \tau_e$ and $\xi \cdot (\ell_v' - \ell_u') > 0$ for $\xi \in (0, \alpha)$. In both cases we get that $\ell_v(\phi+\xi) - \tau_e > \ell_u(\phi+\xi)$ for $\xi \in (0, \alpha)$. It follows with the monotonicity of F_e^+ that

$$z_e(\ell_v(\phi+\xi)) \overset{(6.16)}{=} F_e^+(\ell_v(\phi+\xi)-\tau_e) - F_e^+(\ell_u(\phi+\xi)) \geq F_e^+(\ell_u(\phi+\xi)+\varepsilon) - F_e^+(\ell_u(\phi+\xi)) = \varepsilon \cdot \frac{x_e'}{\ell_u'} > 0,$$

where we choose $\varepsilon > 0$, such that $\ell_u(\phi+\xi) + \varepsilon < \min\{\, \ell_u(\phi+\alpha), \ell_v(\phi+\xi) - \tau_e \,\}$. Note that, since a flow of x_e' leaves node u, there either has to be some inflow of x' into u or $u = s$. In both cases we have $\ell_u' > 0$, and thus, $\ell_u(\phi+\xi) < \ell_u(\phi+\alpha)$ and $\frac{x_e'}{\ell_u'}$ is well-defined. Finally, we get that $b_e^-(\ell_v(\phi+\xi)) = \nu_e^- \geq c_v \cdot \nu_e^-$.

Inflow condition and no-slack condition. For all $\xi \in [0, \alpha)$ we show that

$$f_e^+(\ell_u(\phi+\xi)) \leq b_e^+(\ell_u(\phi+\xi))$$

and that this holds with equality for at least one arc $e \in \delta_v^+$, whenever there is an incoming throttled arc. Equation (6.11) ensures that an arc $e \notin \bar{E}_\phi$ stays non-full during $[\ell_u(\phi), \ell_u(\phi+\alpha))$. Together with (6.10) we get that $b_e^+(\ell_u(\phi+\xi)) = b_e^+$ for all $\xi \in [0, \alpha)$, and hence, (sTF4) yields

$$f_e^+(\ell_u(\phi+\xi)) = \frac{x_e'}{\ell_u'} \leq b_e^+ = b_e^+(\ell_u(\phi+\xi)).$$

An incoming throttled arc implies $c_u < 1$, and thus, the inequality holds with equality in this case due to (sTF5).

No-deadlock condition. Suppose for contradiction that there is a point in time ξ, for which the set of arcs e with $\eta_e = 0$ and $d_e(\xi) = \sigma_e$ contains a cycle $v_1, \dots, v_k = v_0$. For every $i = 0, 1, \dots, k$ we consider the particle with minimal value ϕ_i such that $\ell_{v_i}(\phi_i) = \xi$. By Lemma 6.6 (iv) we have

$$\ell_{v_i}(\phi_{i-1}) - \tau_{v_{i-1}v_i} \geq \ell_{v_{i-1}}(\phi_{i-1}) = \xi = \ell_{v_i}(\phi_i)$$

for every $i = 1, \dots, k$, which implies $\phi_{i-1} \geq \phi_i$. Since the sum of transit times in each cycle is strictly positive there has to be an i with $\tau_{v_{i-1}v_i} > 0$, and therefore $\phi_{i-1} > \phi_i$, which leads to a contradiction.

Earliest arrival times. We show that the extended ℓ-labels fulfill Equations (6.3) and (6.4), and therefore describe the earliest arrival times.
We have $\ell_s' = \frac{1}{c_s \cdot R}$, implying

$$\int_0^{\ell_s(\phi+\xi)} r(\zeta)\,\mathrm{d}\zeta = \phi + \int_{\ell_s(\phi)}^{\ell_s(\phi+\xi)} c_s \cdot R\,\mathrm{d}\zeta = \phi + (\ell_s(\phi+\xi) - \ell_s(\phi)) \cdot \frac{1}{\ell_s'} = \phi + \xi$$

for all $\xi \in [0, \alpha)$. Since $r(\zeta)$ is always strictly positive, $\ell_s(\phi+\xi)$ is also the minimal value that satisfies this equation, and hence, fulfills (6.3).
Considering $v \neq s$, $e = uv \in E$, and $\xi \in [0, \alpha)$, we distinguish two cases and show that $\ell_v(\phi+\xi) \leq T_e(\ell_u(\phi+\xi))$ in the first case and that $\ell_v(\phi+\xi) = T_e(\ell_u(\phi+\xi))$ in the second case.

Case 1: $e \in E \backslash E'_\phi$ or ($e \in E'_\phi \backslash E^*_\phi$ with $\ell'_v < \ell'_u$).
Here, we have for all $\xi \in [0, \alpha)$ that

$$\ell_v(\phi + \xi) \leq \ell_u(\phi) + \tau_e + \xi \cdot \ell'_u \leq T_e(\ell_u(\phi) + \xi \cdot \ell'_u) = T_e(\ell_u(\phi + \xi)),$$

where the first inequality follows either by (6.9) for $e \in E \backslash E'_\phi$ or by $\ell_v(\phi) = \ell_u(\phi) + \tau_e$ and $\ell'_v < \ell'_u$ otherwise.

Case 2: $e \in E^*_\phi$ or ($e \in E'_\phi \backslash E^*_\phi$ with $\ell'_v \geq \ell'_u$).
If $x'_e = 0$ and $e \in E^*_\phi$ we get from (sTF2) that $\ell'_v = \rho_e(\ell'_u, x'_e, c_v) = 0$. Since e is active for ϕ we have

$$\ell_v(\phi + \xi) = \ell_v(\phi) = T_e(\ell_u(\phi)) \leq T_e(\ell_u(\phi + \xi)).$$

To show equality note that with (6.8) we obtain

$$q_e(\ell_u(\phi + \xi)) = T_e(\ell_u(\phi + \xi)) - \ell_u(\phi + \xi) - \tau_e > T_e(\ell_u(\phi + \xi)) - \ell_v(\phi + \xi) \geq 0.$$

Thus, Lemma 6.3 (i) and (v) together with $F^+_e(\ell_u(\phi + \xi)) - F^+_e(\ell_u(\phi)) = \xi \cdot x'_e = 0$ implies $T_e(\ell_u(\phi)) = T_e(\ell_u(\phi + \xi))$.
If we have $x'_e = 0$ and $e \in E'_\phi \backslash E^*_\phi$ with $\ell'_v \geq \ell'_u$ we obtain that $\ell'_v \leq \rho_e(\ell'_u, x'_e, c_v) = \ell'_u \leq \ell'_v$, and hence $\ell'_v = \ell'_u$. This yields

$$\ell_v(\phi + \xi) = \ell_u(\phi) + \tau_e + \xi \cdot \ell'_u = \ell_u(\phi + \xi) + \tau_e = T_e(\ell_u(\phi + \xi)).$$

The last equation holds since the inflow is 0 within $(\ell_u(\phi), \ell_u(\phi + \xi))$, and hence, there is no queue.
Now, suppose that $x'_e > 0$, which implies $\ell'_v = \rho_e(\ell'_u, x'_e, c_v) > 0$. For all $\xi \in [0, \alpha)$ we have

$$\ell_u(\phi + \xi) + \tau_e = \ell_u(\phi) + \tau_e + \xi \cdot \ell'_u \leq \ell_v(\phi) + \xi \cdot \ell'_v = \ell_v(\phi + \xi). \tag{6.17}$$

The inequality follows either by (6.8) in the case of $e \in E^*_\phi$ or, in the other case, by $\ell_v(\phi) = \ell_u(\phi) + \tau_e$ and $\ell'_u \leq \ell'_v$. By definition of q_e and z_e we obtain that $q_e(\ell_u(\phi + \xi))$ is the minimal non-negative value with

$$F^-_e(\ell_u(\phi + \xi) + \tau_e + q_e(\ell_u(\phi + \xi))) = F^+_e(\ell_u(\phi + \xi)) \overset{(6.16)}{=} F^-_e(\ell_v(\phi + \xi)).$$

Note that F^-_e is monotone and strictly increasing at $\ell_v(\phi + \xi)$ with slope $f^-_e(\ell_v(\phi + \xi)) = \frac{x'_e}{\ell'_v} > 0$. This and (6.17) imply that $q_e(\ell_u(\phi + \xi))$ satisfies

$$T_e(\ell_u(\phi + \xi)) = \ell_u(\phi + \xi) + \tau_e + q_e(\ell_u(\phi + \xi)) = \ell_v(\phi + \xi).$$

In conclusion, both cases together show that for all $v \in V \backslash \{ s \}$ and all $\xi \in [0, \alpha)$ we have

$$\ell_v(\phi + \xi) \leq \min_{e = uv \in E} T_e(\ell_u(\phi + \xi)).$$

In order to show equality, recall that (sTF2) yields an arc $e = uv \in E'$ with $\ell'_v = \rho_e(\ell'_u, x'_e, c_v)$. Hence, either $e \in E^*$ or $\ell'_v \geq \ell'_u$, meaning that e belongs to the second case where we have shown equality.

Nash flow condition. Since all conditions are fulfilled, we have a feasible flow over time. The Nash flow condition follows immediately by Lemma 3.3 (iii) and (6.16). Note that by construction, the condition holds for every point in time and not only for almost every point in time. □

7 Instantaneous Dynamic Equilibria

One essential aspect of a Nash equilibrium is the assumption that each player has access to all information of the game and sticks with his or her strategy even after the strategies of the other players have been revealed. In other words, players assume that the other participants act completely rational. Transferred to dynamic equilibria that means that every flow particle in a Nash flow over time does not only have access to all information on the current state of the network but it also predicts the future evolution perfectly, under the assumption that the flow in front uses current shortest paths only. While this might be a typical assumption in mathematical game theory, it is far from realistic in today's traffic. Even though the routing data of the majority of traffic users might be available in the near future (of course it is up to discussion if this is desirable society-wise) there will always be a factor of uncertainty, since we cannot assume complete rationality of all traffic users.

On the one hand, one can argue that a Nash flow over time is a state of traffic that evolves after multiple rounds. This makes sense when considering the daily commuters facing more or less the same setting every day. They might take some highly congested routes with the smallest free flow transit time at the beginning, but they will deviate to quicker alternatives after some days of experience. On the other hand, Nash flows over time neither incorporate spontaneous changes of the network, such as car accidents, nor do they allow for instantaneous route changes of the agents. To address these issues we want to discuss a different class of flows over time, called **instantaneous dynamic equilibria**, or **IDE flows** for short, in this chapter. Introduced by Lukas Graf and Tobias Harks in early 2019 [36], these IDE flows are much more realistic concerning some aspects of today's traffic, but they also have their own drawbacks and oddities.

The main idea is the following. We consider a feasible flow over time in a multi-commodity setting where every particle starts at some origin and heads to some destination. But instead of deciding on complete origin-destination paths at the very beginning, particles only choose the first link in the right direction, while having a complete route in mind. When reaching the next junction, every particle reconsiders its choice and might adopt its path to the destination. Importantly, this decision is only based on the current configuration of the network as particles do not anticipate the future evolution. In other words, at every node each particle chooses a route with the shortest travel time to the destination with respect to the waiting times at the exact moment of the decision. Obviously, this is well motivated by real-world traffic, as we might assume that each road user follows a navigation system that reports the delays on every road in the network in real-time and he or she decides at every crossing to choose a path that has the shortest travel time to the destination at that very moment. Of course this route might not be chosen wisely as the network can change drastically over time. In fact, it is even possible that particles go back to a location they have already visited before. An example where this occurs is given in Figure 7.1. But this is not even the worst that can happen. By a careful construction of a network it is possible to create an IDE flow where no particle will *ever* reach its destination, but instead, they cycle around the network for eternity. This special network construction as well as the existence of IDE flows in networks with only a single destination were shown by Graf and Harks in their first publication on this topic [36].

In this chapter, however, we will focus on the structure and existence of IDE flows in the multi-origin and multi-destination setting. This theory was developed in collaboration with Graf and Harks [37]. It turns out that these multi-commodity IDE flows have a lot in common with single-commodity Nash flows over time. They can even be constructed by a sequence of thin-flow-like objects as we will show in the following.

7.1 Flow dynamics

We consider the multi-commodity flow setting as it is introduced in Section 5.1.1 from page 70 onwards. We will briefly recall the central definitions.
In this chapter a network consists of a directed graph $G = (V, E)$, where every arc e is equipped with a transit time $\tau_e > 0$ and a capacity $\nu_e > 0$, and a finite set of commodities J, each with an origin-destination pair (s_j, t_j) as well as a right-constant network inflow rate function $r_j\colon [0,\infty) \to [0,\infty)$. We denote the set of nodes that can reach t_j by V_j and we assume that $s_j \in V_j$.

Multi-commodity flows over time. As before we represent a flow over time by a family of locally integrable and bounded functions $f = (f^+_{j,e}, f^-_{j,e})_{j\in J, e\in E}$, describing the **in-** and **outflow rate** functions of each commodity. The **cumulative in-** and **outflow** is denoted by $F^+_{j,e}$ and $F^-_{j,e}$. Furthermore, the **total flow rates** and **total cumulative flows** of all commodities combined are defined for all $\theta \in [0,\infty)$ by

$$f^+_e(\theta) := \sum_{j\in J} f^+_{j,e}(\theta), \quad f^-_e(\theta) := \sum_{j\in J} f^-_{j,e}(\theta), \quad F^+_e(\theta) := \sum_{j\in J} F^+_{j,e}(\theta) \quad \text{and} \quad F^-_e(\theta) := \sum_{j\in J} F^-_{j,e}(\theta).$$

Flow conservation. We say that f is a **multi-commodity flow over time** if it satisfies the following flow conservation conditions:

$$F^-_{j,e}(\theta+\tau_e) \leq F^+_{j,e}(\theta) \qquad \text{for all } \theta \in [0,\infty), \tag{7.1}$$

$$\sum_{e\in\delta^+_v} f^+_{j,e}(\theta) - \sum_{e\in\delta^-_v} f^-_{j,e}(\theta) = \begin{cases} 0 & \text{if } v \in V\setminus\{\, s_j \,\}, \\ r_j(\theta) & \text{if } v = s_j, \end{cases} \qquad \text{for almost all } \theta\in[0,\infty). \tag{7.2}$$

Queues, waiting times and exit times. The queue sizes, waiting times and exit times are defined by

$$z_e(\theta) := F^+_e(\theta-\tau_e) - F^-_e(\theta), \qquad q_e(\theta) := \frac{z_e(\theta+\tau_e)}{\nu_e} \quad \text{and} \quad T_e(\theta) := \theta + \tau_e + q_e(\theta).$$

Feasibility. We call a multi-commodity flow over time f **feasible** if the total outflow rates satisfy

$$f^-_e(\theta) = \begin{cases} \nu_e & \text{if } z_e(\theta) > 0, \\ \min\{\, f^+_e(\theta-\tau_e), \nu_e \,\} & \text{if } z_e(\theta) = 0, \end{cases} \qquad \text{for almost all } \theta\in[0,\infty) \tag{7.3}$$

and if the outflow rate of every commodity $j \in J$ individually fulfills

$$f^-_{j,e}(\theta) = \begin{cases} f^-_e(\theta)\cdot\frac{f^+_{j,e}(\vartheta)}{f^+_e(\vartheta)} & \text{if } f^+_e(\vartheta) > 0, \\ 0 & \text{else,} \end{cases} \qquad \text{for almost all } \theta\in[0,\infty). \tag{7.4}$$

Here, $\vartheta = \min\{\, \xi \leq \theta \mid T_e(\xi) = \theta \,\}$ denotes the earliest point in time a particle can enter the arc e in order to leave the queue at time θ.

7.2 Instantaneous Dynamic Equilibria

The most essential information for a particle in an IDE flow is not the earliest time to arrive at some node, but the fastest way to reach the destination from its current location.

Current shortest path distances. Hence, we denote for every commodity j the **current shortest path distance** from a node $v \in V_j$ to t_j by a function $\ell_{j,v} \colon [0, \infty) \to [0, \infty)$ defined by

$$\ell_{j,v}(\theta) := \begin{cases} 0 & \text{for } v = t_j, \\ \min\limits_{e=vw \in E} \tau_e + q_e(\theta) + \ell_{j,w}(\theta) & \text{else.} \end{cases} \tag{7.5}$$

Clearly, this is well-defined as all transit times are strictly positive. Furthermore, since q_e is almost everywhere differentiable so are these ℓ-functions.
We say that an arc $e = vw$ with $w \in V_j$ is **active** for $j \in J$ at time θ, if

$$\ell_{j,v}(\theta) = \tau_e + q_e(\theta) + \ell_{j,w}(\theta).$$

The set of active arcs is denoted by $E'_{j,\theta}$ and we call the graph $G'_{j,\theta} := (V_j, E'_{j,\theta})$ the **current shortest paths network**. Note that in contrast to the earliest arrival times for Nash flows over time, we always consider the waiting time at θ even for arcs further ahead. Hence, when considering an active path P from some node v to t_j we have

$$\ell_{j,v}(\theta) = \sum_{e \in P} (\tau_e + q_e(\theta)).$$

Queuing arcs. We call the arcs that have a positive waiting time for particles entering at time θ the **queuing arcs** and denote them by

$$E^*_\theta := \{\, e \in E \mid q_e(\theta) > 0 \,\}.$$

Note that this set is not commodity specific. However, the arcs in $E^*_\theta \cap E'_{j,\theta}$ are comparable to the resetting arcs of Nash flows over time.

IDE flows. As motivated at the beginning of this chapter, we want to consider flows over time where every particle only uses arcs that lie on a current shortest path to the destination. In other words, flow should only enter into active arcs. This leads to the following definition.

Definition 7.1 (Instantaneous dynamic equilibrium).
A feasible flow over time f is an **instantaneous dynamic equilibrium** (**IDE flow** for short), if for all $j \in J$ and every $e \in E$ the **IDE condition** is satisfied for almost every point in time θ:

$$f^+_{i,e}(\theta) > 0 \quad \Rightarrow \quad e \in E'_{j,\theta}. \tag{IDE}$$

The next sections are dedicated to the proof of the existence of these multi-commodity IDE flows. Similar to the existence proof for Nash flows over time, we will start with the empty flow over time and extend it step by step. This time, however, we do not consider the particles in order of their source arrival time. Instead, for a given snapshot time θ, we consider the incoming flow at all nodes simultaneously. In other words, we consider the same point in time at all nodes at once.
Note that it is easy for one specific particle at node v at one specific point in time θ to determine a current shortest path to the particle's destination. The main difficulty is to distribute all particles

leaving node v during a whole interval $[\theta, \theta+\alpha)$, because the flow needs to split up in such a way that all outgoing arcs that are used stay active during the whole interval. Of course, this does not only depend on the amount of flow that is sent into this very arc, but also on the change of waiting times on all arcs on all current shortest paths to the destination. In networks where all commodities have the same destination this can easily be done by considering the nodes one by one, starting with the ones closest to the common destination. For multiple destinations, however, this is a bit more involved since we need to consider all outgoing flows of all nodes simultaneously.

Throughput rates. Given a feasible flow over time f we want to formulate necessary conditions (which are also sufficient, as we will see later) for f to be an IDE flow. For this we denote the **throughput rate** of a commodity j at node v at time θ by $b_{j,v}(\theta)$. More precisely, we define for every $j \in J$ and $v \in V$

$$b_{j,v}(\theta) := \begin{cases} \sum_{e \in \delta_v^-} f_{j,e}^-(\theta) & \text{for } v \in V \setminus \{ s_j \}, \\ \sum_{e \in \delta_v^-} f_{j,e}^-(\theta) + r_j(\theta) & \text{for } v = s_j. \end{cases}$$

7.3 Thin Flows for Instantaneous Dynamic Equilibria

Similar to the thin flows with resetting for Nash flows over time, we define a set of conditions on the derivatives of the current shortest path distances $(\ell'_{j,v})_{j \in J, v \in V}$ and the outflow rate, which we will denote by $(x'_{j,e})_{j \in J, e \in E}$ (in the style of the thin flows for Nash flows over time).

Definition 7.2 (IDE thin flow).
Given a directed graph $G = (V, E)$ and a set of commodities J with destinations t_j for all $j \in J$, we consider non-negative throughput rates $(b_{j,v})_{j \in J, v \in V}$ and a set of arcs with queues $E^* \subseteq E$ as well as a current shortest paths network $G'_j = (V_j, E'_j)$ for every commodity j. Suppose G'_j is acyclic and every node v with $b_{j,v} > 0$ is in V_j and can reach t_j within G'_j. We call the pair of two real vectors $(x'_{j,e})_{j \in J, e \in E}$ and $(\ell'_{j,v})_{j \in J, v \in V_j}$ an **IDE thin flow** if the following equations hold:

$$\sum_{vw \in E} x'_{j,e} = b_{j,v} \qquad \text{for all } j \in J \text{ and } v \in V \setminus \{ t_j \}, \tag{ideTF1}$$

$$x'_{j,e} = 0 \qquad \text{for all } j \in J \text{ and } e \in E \setminus E'_j, \tag{ideTF2}$$

$$\ell'_{j,t_j} = 0 \qquad \text{for all } j \in J, \tag{ideTF3}$$

$$\ell'_{j,v} = \min_{e = vw \in E'_j} \rho_e\left(\textstyle\sum_{i \in J} x'_{i,e}\right) + \ell'_{j,w} \qquad \text{for all } j \in J \text{ and } v \in V_j \setminus \{ t_j \}, \tag{ideTF4}$$

$$\ell'_{j,v} = \rho_e\left(\textstyle\sum_{i \in J} x'_{i,e}\right) + \ell'_{j,w} \qquad \text{for all } j \in J \text{ and } e = vw \in E'_j \text{ with } x'_{j,e} > 0, \tag{ideTF5}$$

where

$$\rho_e(x'_e) := \begin{cases} \frac{x'_e}{\nu_e} - 1 & \text{if } e \in E^*, \\ \max\left\{ \frac{x'_e}{\nu_e} - 1, 0 \right\} & \text{if } e \in E \setminus E^*. \end{cases}$$

Note that this time $(x'_{j,e})_{e \in E}$ does not form a static flow in general, as the throughput rate $b_{j,v}(\theta)$ at a node v is not directly related to the throughput rate $b_{j,w}(\theta)$ at another node w. In other words, if we have an arc $e = vw$ and the thin flow sends an inflow rate of $x'_{j,e}$ into that arc, this flow will arrive at w at time $T_e(\theta) > \theta$, which will only be relevant for a later thin flow phase.

Furthermore, the function ρ_e maps the total inflow x'_e to the change of the waiting time q_e. In other words, if the total inflow rate exceeds the capacity, a queue builds up with a rate of $x'_e - \nu_e$, and therefore, the waiting time increases with a slope of $\frac{x'_e}{\nu_e} - 1$. Analogously, if the total inflow rate is less than the capacity, the waiting time decreases with a slope of $\frac{x'_e}{\nu_e} - 1$ if there was a queue to begin with, or stays 0 otherwise.
With this observations it is easy to see that (ideTF4) describes the change of the current shortest path distance of node v and commodity j. Considering an arc $e = vw$ the travel time of the arc itself changes with a slope given by ρ_e in dependency of the total inflow rate into this arc. In addition, the change of travel time from w to t_j is given by $\ell'_{j,w}$. Hence, the slope of $\ell_{j,v}$ is given by the sum of these two values.
Since we consider the current shortest paths network at time θ, all outgoing arcs $vw \in E'_j$ are active at the beginning. But only the arcs that attain the minimum in (ideTF4) stay active during the phase, and therefore, only these can be used by an IDE flow. This intuition leads to the following theorem.

Theorem 7.3.

For almost all $\theta \in [0, \infty)$ the inflow rate $x' = (f^+_{j,e}(\theta))_{j \in J, e \in E}$ of an IDE flow f, together with the derivatives of the current shortest path distances $(\ell'_{j,v})_{j \in J, v \in V}$, forms an IDE thin flow with throughput rates $(b_{j,v}(\theta))_{j \in J, v \in V}$, active arcs $(E'_{j,\theta})_{j \in J}$ and waiting arcs E^*_θ.

Proof. Let $\theta \in [0, \infty)$ be a point in time where all $\ell'_{j,v}$ are differentiable, where flow conservation (3.2), the feasibility condition (7.3) as well as the IDE condition (IDE) hold and where $\frac{\mathrm{d}}{\mathrm{d}\theta} F^+_{j,e}(\theta) = f^+_{j,e}(\theta)$ for all $j \in J$ and all $e \in E$. This is given for almost all $\theta \in [0, \infty)$.
We have that each node v with $b_{j,v}(\theta) > 0$ is contained in V_j, and therefore, v can reach t_j within the acyclic current shortest paths network $G'_{j,\theta}$. Equation (ideTF1) follows immediately by the flow conservation condition, (ideTF2) by the IDE flow condition and (ideTF3) by the definition of $\ell_{j,t_j}(\theta) = 0$.
For the remaining two equations we consider the derivatives of (7.5). By the differentiation rule for a minimum (Lemma 2.3 on page 17) we have

$$\ell'_{j,v}(\theta) = \min_{e = vw \in E'_{j,\theta}} q'_e(\theta) + \ell'_{j,w}(\theta).$$

By Lemma 3.1 (vii) applied to the total inflow rate $f^+_e(\theta)$ we have that

$$q'_e(\theta) = \left\{ \begin{array}{ll} \frac{f^+_e(\theta)}{\nu_e} - 1 & \text{if } q_e(\theta) > 0, \\ \max\left\{ \frac{f^+_e(\theta)}{\nu_e} - 1, 0 \right\} & \text{else,} \end{array} \right\} = \rho_e\left(\sum_{i \in J} f^+_{i,e}(\theta) \right),$$

which shows (ideTF4).
Finally, for (ideTF5) suppose for contradiction that

$$\ell'_{j,v}(\theta) < \rho_e\left(\sum_{i \in J} f^+_{i,e}(\theta) \right) + \ell'_{j,w}(\theta) = q'_e(\theta) + \ell'_{j,w}(\theta)$$

for some arc $e \in E'_{j,\theta}$ with $f^+_{j,e}(\theta) > 0$. But this contradicts the IDE condition for the times in $(\theta, \theta + \varepsilon)$ for some small $\varepsilon > 0$, since e will immediately leave the current shortest paths network but the inflow rate is strictly positive during this interval. Hence, (ideTF5) is satisfied, which finishes the proof. $\square$

As the next step, we will show that these IDE thin flows always exist for all throughput rates $b_{j,e}$ and all active and queuing arcs that satisfy the requirements.

Theorem 7.4.

For all current shortest paths networks $G'_j = (V_j, E'_j)$, arc subsets E^* and non-negative throughput rates $(b^-_{j,v})_{j\in J, v\in V}$, such that G'_j is acyclic and every node v with $b_{j,v} > 0$ is in V_j and reaches t_j within G'_j, there exists an IDE thin flow (x', ℓ').

Proof. For every vector x' satisfying (ideTF1) and (ideTF2) we can define unique node labels ℓ' that fulfill (ideTF3) and (ideTF4). Furthermore, this mapping $x' \mapsto \ell'$ is continuous. Here, existence follows since E'_j is acyclic and the uniqueness follows from the fact that for every node v there is a v-t_j-path within E'_j. Thus, the only difficult part is to show (ideTF5), which we will do once more with the help of Kakutani's fixed point theorem (Theorem 2.9 on page 22).
In order to do so let X be the set of x' vectors that satisfy (ideTF1) and (ideTF2), i.e.,

$$X := \left\{ x' \in \mathbb{R}^{J\times E}_{\geq 0} \;\middle|\; \begin{array}{ll} \sum_{e\in\delta^+_v} x'_{j,e} = b_{j,v} & \text{for all } j \in J \text{ and } v \in V \setminus \{\, t_j \,\} \\ \text{and } \; x'_{j,e} = 0 & \text{for all } j \in J \text{ and } e \in E \setminus E'_j \end{array} \right\}.$$

Clearly, X is compact, convex and non-empty. We define a set-valued function $\Gamma : X \to 2^X$ as follows:

$$\Gamma(x') := \left\{\, y \in X : y_{j,e} = 0 \text{ for all } e \in E'_j \text{ with } \ell'_{j,v} < \rho_e\left(\textstyle\sum_{i\in J} x'_{i,e}\right) + \ell'_{j,w} \,\right\}$$

where ℓ' are the labels corresponding to x'. Then, $\Gamma(x)$ is non-empty and convex. For non-emptiness note that every node in V_j, except for t_j, has at least one outgoing arc with $\ell'_{j,v} = \rho_e\left(\sum_{i\in J} x'_{i,e}\right) + \ell'_{j,w}$, so y can send everything there. Convexity is clear as well, since x' determines which arcs can be used and those are fixed within $\Gamma(x')$.
Finally, in order to apply Kakutani's fixed point theorem we show that $\{\, (x, y) \mid y \in \Gamma(x) \,\}$ is a closed set. Let $(x^n, y^n)_{n\in\mathbb{N}}$ be a sequence within this set that converges in $\mathbb{R}^{J\times E} \times \mathbb{R}^{J\times E}$. Since X is compact, both sequences separately converge to some point x^* and y^*. Let $(\ell^n)_{n\in\mathbb{N}}$ be the sequence of associated node labels of x^n and ℓ^* the node labels of x^*. Since $x \mapsto \ell'$ is continuous we have $\ell^* = \lim_{n\to\infty} \ell^n$.
Finally, we need to show that $y^* \in \Gamma(x^*)$. Suppose there is a commodity $j \in J$ and an arc $e = vw \in E'_j$ with $y^*_{j,e} > 0$ and $\ell^*_{j,v} < \rho_e\left(\sum_{i\in J} x^*_{i,e}\right) + \ell^*_{j,w}$. But since ρ_e is continuous, there has to be an $n_0 \in \mathbb{N}$ such that $y^n_{j,e} > 0$ and $\ell^n_{j,v} < \rho_e\left(\sum_{i\in J} x^n_{i,e}\right) + \ell^n_{j,w}$ for all $n \geq n_0$. This is a contradiction to $y^n \in \Gamma(x^n)$.
With Kakutani's fixed point theorem (Theorem 2.9) there exists an $x'_* \in X$ with $x'_* \in \Gamma(x'_*)$, which forms, together with the associated node labels ℓ'_*, an IDE thin flow. □

7.4 Constructing Instantaneous Dynamic Equilibria

Similar to the construction of Nash flows over time, we want to use IDE thin flows to extend an IDE flow step by step.

α-Extensions. Consider an IDE flow f where the inflow rates $f^+_{j,e}$ are already defined for all times in $[0, \theta)$ and are right-constant. We call f an **IDE flow up to time** θ. Note that due to continuity of q_e and $\ell_{j,v}$ we can determine the current shortest paths networks $G'_{j,\theta} = (V_j, E'_{j,\theta})$ as well as the throughput rates $b_{j,v}(\theta)$ since the feasibility conditions (7.3) and (7.4) determine unique outflow rates $f^-_{j,e}(\theta)$ for given inflow rates $f^+_{j,e}$ from the past.

In order to extend f we consider an IDE thin flow (x', ℓ') and extend the inflow rates and ℓ-labels for all $j \in J$, $e \in E$ and $v \in V_j$ by

$$f^+_{j,e}(\theta + \xi) := x'_{j,e} \quad \text{and} \quad \ell_{j,v}(\theta + \xi) := \ell_{j,v}(\theta) + \xi \cdot \ell'_{j,v} \qquad \text{for all } \xi \in [0, \alpha).$$

We call this extended flow over time α-**extension**.

Feasible extension steps and thin flow phases. To ensure that we end up with an IDE flow up to time $\theta + \alpha$ the following requirements on the **extension step size** α must be satisfied.
First of all, the waiting time can never be negative, and therefore, the phase ends as soon as a queue depletes:

$$q_e(\theta) + \alpha \cdot \left(\frac{\sum_{i \in J} x'_{i,e}}{\nu_e} - 1 \right) \geq 0 \qquad \text{for all } e \in E^*_\theta. \tag{7.6}$$

Furthermore, the phase ends as soon as an inactive arc gets active. Since queues can also grow on inactive arcs (due to flow of other commodities) we need to take the changing rate of a queue into account. Hence, for all $j \in J$ and $e = vw \in E \setminus E'_{j,\theta}$ with $v, w \in V_j$ we require that

$$\ell_{j,v}(\theta) + \alpha \cdot \ell'_{j,v} \leq \tau_e + q_e(\theta) + \alpha \cdot \rho_e \left(\textstyle\sum_{i \in J} x'_{i,e} \right) + \ell_{j,w}(\theta) + \alpha \cdot \ell'_{j,w}. \tag{7.7}$$

Finally, the throughput rate should stay constant during a phase:

$$b_{j,v}(\theta + \xi) = b_{j,v}(\theta) \qquad \text{for all } j \in J \text{ and } v \in V \setminus \{ t_j \} \text{ and all } \xi \in [0, \alpha). \tag{7.8}$$

We call $\alpha > 0$ **feasible** if it satisfies (7.6), (7.7) and (7.8). It is easy to see that such a feasible $\alpha > 0$ always exists, since $q_e(\theta) > 0$ for all $e \in E^*_\theta$ and $\ell_{j,v}(\theta) < \tau_e + q_e(\theta) + \ell_{j,w}(\theta)$ for all $j \in J$ and $e = vw \in E \setminus E'_{j,\theta}$ as well as $v, w \in V_j$. Furthermore, the functions $b_{j,v}$ are right-constant, since $f^+_{j,e}$ as well as r_j are right-constant. Since $\tau_e > 0$ for all $e \in E$ we have that $b_{j,v}(\theta)$ is well-defined and constant on some small interval $[\theta, \theta + \varepsilon)$.
For the maximal α we call the interval $[\theta, \theta + \alpha)$ a **thin flow phase**.

Theorem 7.5.
Consider an IDE flow up to time θ, an IDE thin flow (x, ℓ') at time θ and a feasible $\alpha > 0$. The α-extension is an IDE flow up to time $\theta + \alpha$ and the extended ℓ-functions denote the current shortest path distances.

Proof. First note that the feasibility conditions are satisfied since the outflow rates $f^-_{j,e}$ are defined exactly that way. Furthermore, flow conservation holds at all nodes $v \in V \setminus \{ t_j \}$ and for every commodity $j \in J$ since for all $\xi \in [0, \alpha)$ we have

$$\sum_{e \in \delta^+_v} f^+_{j,e}(\theta + \xi) = \sum_{e \in \delta^+_v} x'_{j,e} = b_{j,v}(\theta) = b_{j,v}(\theta + \xi) = \begin{cases} \sum_{e \in \delta^-_v} f^-_{j,e}(\theta + \xi) & \text{for } v \neq s_j, \\ \sum_{e \in \delta^-_v} f^-_{j,e}(\theta + \xi) + r_j(\theta + \xi) & \text{for } v = s_j. \end{cases}$$

Next, we show that the ℓ labels satisfy Equation (7.5). Given a point in time $\theta + \xi$ with $\xi \in [0, \alpha)$ we have by Lemma 3.1 (vii) applied to the total inflow rate $f^+_e(\theta + \xi)$ that

$$q'_e(\theta + \xi) = \left\{ \begin{array}{ll} \frac{f^+_e(\theta+\xi)}{\nu_e} - 1 & \text{if } q_e(\theta + \xi) > 0, \\ \max\left\{ \frac{f^+_e(\theta+\xi)}{\nu_e} - 1, 0 \right\} & \text{else,} \end{array} \right\} = \rho_e \left(\textstyle\sum_{i \in J} x'_{i,e} \right).$$

Note that $q'_e(\theta + \xi)$ is constant during the thin flow phase, i.e., $q_e(\theta + \xi) = q_e(\theta) + \xi \cdot \rho_e \left(\sum_{i \in J} x'_{i,e} \right)$.

For non-active arcs $e = vw \notin E'_{j,\theta}$ with $v, w \in V_j$ we have by (7.7) that

$$\ell_{j,v}(\theta+\xi) = \ell_{j,v}(\theta)+\xi\cdot\ell'_{j,v} \leq \tau_e+q_e(\theta)+\xi\cdot\rho_e\left(\sum_{i\in J} x'_{i,e}\right)+\ell_{j,w}(\theta)+\xi\cdot\ell'_{j,w} = \tau_e+q_e(\theta+\xi)+\ell_{j,w}(\theta+\xi).$$

For active arcs $e = vw \in E'_{j,\theta}$ we have by (ideTF4) that

$$\begin{aligned}\ell_{j,v}(\theta+\xi) = \ell_{j,v}(\theta)+\xi\cdot\ell'_{j,v} &\leq \tau_e+q_e(\theta)+\ell_{j,w}(\theta)+\xi\cdot\left(\rho_e\left(\sum_{i\in J} x'_{i,e}\right)+\ell'_{j,w}\right)\\ &= \tau_e+q_e(\theta+\xi)+\ell_{j,w}(\theta+\xi).\end{aligned}$$

Since there has to be one active arc that satisfies (ideTF4) with equality, the same arc satisfies the inequality above with equality, which shows that (7.5) holds. In other words, the extended ℓ labels denote the current shortest path distances in the α-extension.

Finally, we show that the α-extension satisfies the IDE condition (IDE). For all $\xi \in [0, \alpha)$ and all arcs $e = vw \in E$ we have that

$$\begin{aligned} f^+_{j,e}(\theta+\xi) > 0 \quad &\Rightarrow \quad x'_{j,e} > 0\\ &\Rightarrow \quad \ell'_{j,v}(\theta+\xi) = \ell'_{j,v} \overset{\text{(ideTF5)}}{=} \rho_e\left(\sum_{i\in J} x'_{i,e}\right)+\ell'_{j,w} = q'_e(\theta+\xi)+\ell'_{j,w}(\theta)\\ &\Rightarrow \quad e \in E'_{j,\theta+\xi}.\end{aligned}$$

Hence, the α-extension is indeed an IDE flow up to time $\theta+\alpha$. □

As we are now able to extend IDE flows by some small time interval, we can use an approach similar to the one for Nash flows over time in order to show the existence of an IDE flow for all times.

Theorem 7.6.

For right-constant inflow rate functions, there exists an IDE flow f.

Proof. Let $\mathcal{F}_0$ be the set of tuples (f, θ), where $\theta \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and where f is an IDE flow up to time θ with right-constant functions f^+_e and f^-_e. We define

$$\hat{\theta}_0 := \sup\{\,\theta \geq 0 \mid \text{ there exists an IDE flow } f \text{ such that } (f,\theta) \in \mathcal{F}_0\,\}.$$

If $\hat{\theta}_0 = \infty$ we are done, so suppose $\hat{\theta}_0 < \infty$. There exists an IDE flow f_1 with $\theta_1 := \frac{\hat{\theta}_0}{2}$ such that $(f_1, \theta_1) \in \mathcal{F}_0$. With this we define

$$\mathcal{F}_1 := \left\{ (f,\theta) \in \mathcal{F}_0 \,\middle|\, f\big|_{[0,\theta_1)} = f_1 \right\}.$$

Here, $f\big|_{[0,\theta_1)}$ denotes the flow over time up to time θ_1 that is obtained by restricting all in- and outflow rates to the interval $[0, \theta_1)$. Note that if f is an IDE flow up to time θ and $\theta > \theta_1$ then $f\big|_{[0,\theta_1)}$ is an IDE flow up to time θ_1.

The set $\mathcal{F}_1$ is non-empty, so we set

$$\hat{\theta}_1 := \sup\{\,\theta \geq 0 \mid \text{ there exists an IDE flow } f \text{ such that } (f,\theta) \in \mathcal{F}_1\,\}.$$

By Theorem 7.5 we know that $\hat{\theta}_1 > \theta_1$, and therefore $\hat{\theta}_1 \in (\theta_1, \hat{\theta}_0]$. Let $\theta_2 := \frac{\hat{\theta}_1-\theta_1}{2}$. Continuing this construction, we obtain a strictly increasing sequence $(\theta_i)_{i\in\mathbb{N}}$ and a non-increasing sequence $(\hat{\theta}_i)_{i\in\mathbb{N}}$ with $\theta_i < \hat{\theta}_i$ for all $i \in \mathbb{N}$ and $\hat{\theta}_i - \theta_i \leq \frac{\hat{\theta}_0}{2^i} \to 0$ for $i \to \infty$. Let θ^* be the limit of these two sequences. By taking point-wise limits of the sequence $(f_i)_{i\in\mathbb{N}}$ we can construct an IDE flow f^* such that $(f^*, \theta^*) \in \mathcal{F}_0$. These point-wise limits exist for all $\theta < \theta^*$, since the inflow rates at time θ of f_i are constant with respect to i as soon as i is large enough such that $\theta_i > \theta$.

Again by Theorem 7.5 we can extend f^* by some $\alpha > 0$ but this is a contradiction to the definition of $\hat{\theta}_i$ for all i with $\hat{\theta}_i \in [\theta^*, \theta^* + \alpha)$. Hence, $\hat{\theta}_0 = \infty$, which finishes the proof. $\square$

Example. In Figure 7.1 on the next page we give an example of an IDE flow with three commodities. At the beginning at time 0, the shortest path from s_1 to t_1 is clearly the direct top route (s_1, v_1, s_2, t_1) with a current shortest distance of 9. Hence, all particles of commodity 1 take this path at the start. However, as soon as the first particle of commodity 1 arrives at v_1 at time 3, we observe that commodity 2 has built up a large congestion on arc s_2t_1. For this particle both paths (v_1, s_2, t_1) and $(v_1, v_2, s_3, v_3, t_1)$ now have a current distance of 12. Since the queue on s_2t_1 continues to grow due to commodity 2, the arc v_1s_2 becomes inactive for commodity 1 right after time 3. Thus, all particles of this commodity have to take the bottom route.

At time 4 the first particle of commodity 3 reaches node v_3 and since both outgoing arcs have a capacity restriction, the flow has to split up in order to ensure that both arcs v_3t_1 and v_3t_2 stay active. Hence, a flow rate of 3 enters arc v_3t_1 and the remaining flow rate of 1 takes v_3t_2.

The first particle of commodity 1 arrives at s_3 at time 9 only to recognize that there is now a huge congestion on arc v_3t_1. With the current waiting times it would take $6 + 15 = 21$ time units to stay on this bottom path, whereas the path (s_3, v_1, s_2, t_1) has a current path distance of only 12. In other words, since s_3v_3 is not active anymore, all flow particles of commodity 1 choose to go back to the top route, where they have done a full cycle at time 12. As the queue on s_2t_1 is only decreasing, this time, every flow particle sticks with its choice of the top route until leaving the network.

7.5 Further Results on Instantaneous Dynamic Equilibria

In the following we give a brief overview of further interesting results on IDE flows.

Non-termination. First of all, it is worth noting that IDE flows might not terminate. By this we mean that, even though the network inflow rates stop sending in flow at some point in time, the IDE flow keeps cycling in the network indefinitely. Such a strange behavior is really surprising at first sight and the construction of a network for proving this is quite complicated.

The rough idea is to consider two commodities, each cycling in multiple copies of the same network gadget that consists of two directed cycles with one common arc. In order to force the flow to keep cycling in these gadgets we connect each node along the cycles with the sink by a path that is directed through multiple copies of the gadget of the other commodity. By careful construction it is possible that the waiting times on these paths behave exactly in such a way that whenever the flow reaches a node the current fastest way is to stay in the double-cycle-gadget and to leave it at a node further ahead. But as soon as the flow reaches this node, the waiting times have changed and, again, the current shortest path forces the flow to stay in the gadget. In other words, the flow keeps cycling and builds up a queue on the central arc from time to time. Exactly this arc, with varying waiting times, is used by paths of the other commodity. For more details on this we refer to [36].

Termination for single-destination networks. If all commodities have the same destination an IDE flow essentially becomes a single-commodity flow over time. In this case it is possible to show that the flow always terminates, i.e., that all flow reaches the destination at some point in time if the network inflow rate has finite support. The basic idea to show this is to consider the flow closest to

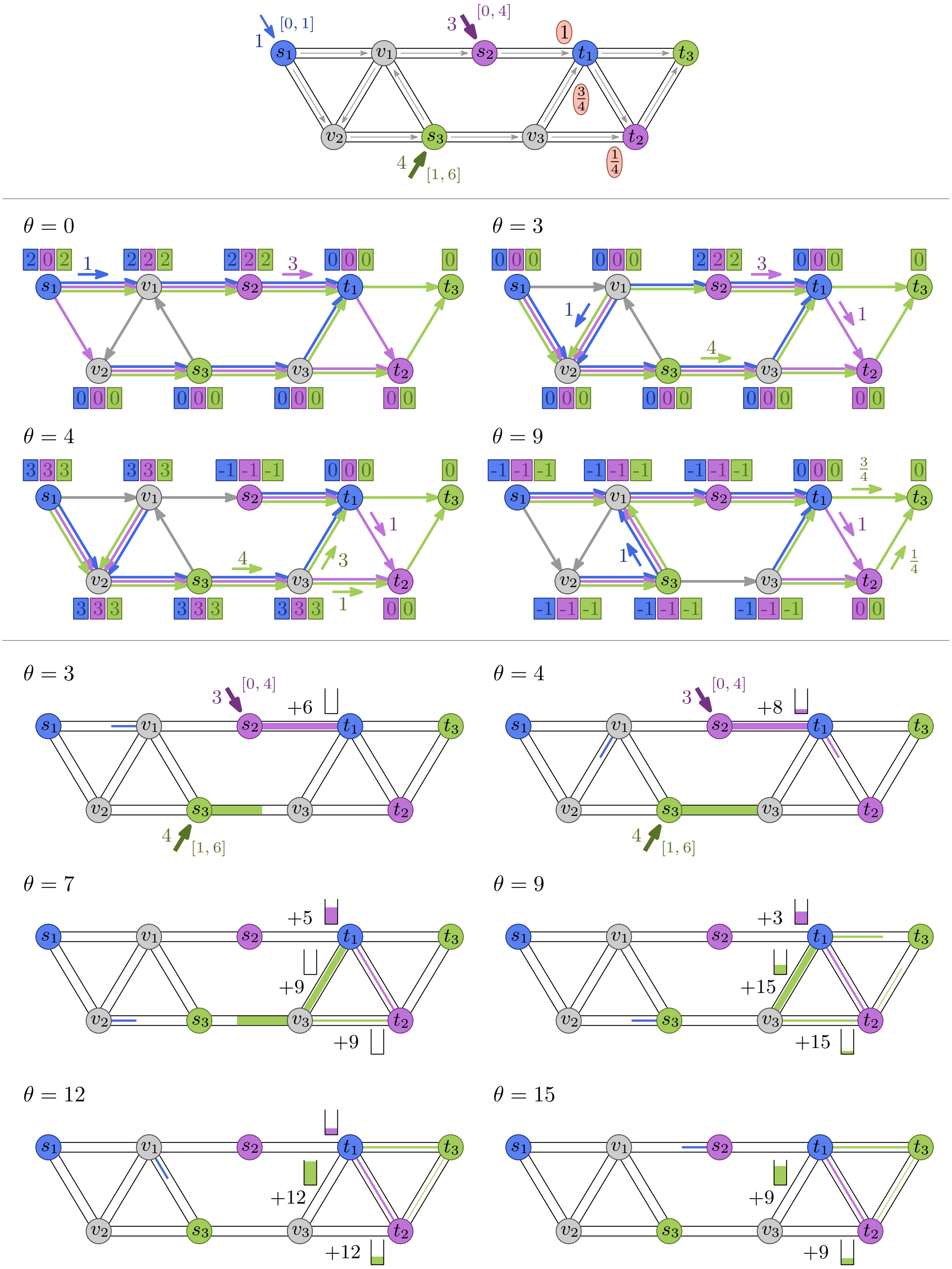


Figure 7.1: An example of an IDE flow with three commodities. The network properties are displayed *on top*. The transit time of all arcs is 3 and all capacities are large enough except for $\nu_{s_2t_1} = 1$, $\nu_{v_3t_1} = \frac{3}{4}$ and $\nu_{v_3t_2} = \frac{1}{4}$. *In the middle* some of the IDE thin flows are depicted at crucial points in time. Arcs that are active for some commodities are shown in the respective colors. Positive thin flow values $x'_{j,e}$ are shown next to each arc and the values in the boxes next to the nodes are the ℓ'-labels of the respective commodity. *In the bottom* we show the flow over time at some snapshots in time. Here, the labels $+q$ indicates a waiting time of q for all particles that enter this arc at this moment in time.

the common destination. This flow will never go in cycles. Instead, it takes the shortest path to the sink according to transit times only and leaves the network there. Again, for details we refer to [36].

Non-uniqueness. As a final remark on IDE flows we want to mention that they are not unique in any sense, not even for a single commodity. To see this consider a network consisting of two parallel s-t-paths $P = (e_1, e_2)$ and $P' = (e'_1, e'_2)$ of equal total transit time. If the first arcs, e_1 and e'_1, of both paths have large capacities (larger than the network inflow rate), then every distribution of flow into e_1 and e'_1 is a valid IDE thin flow at the beginning, since no queue is building up.
But not even the current shortest path distances are unique. Suppose that the second arcs, e_2 and e'_2, on both paths have very small capacities. Then the queues, and therefore the current shortest path distances, heavily depend on the distribution of flow into the paths at node s. But since the particles do not anticipate later stages, the decision mode at s does not depend on the capacities of e_2 or e'_2 at all. Hence, depending on the IDE thin flow at time 0 we end up with different current shortest path distances later on.

Future Research and Conclusion

8

In this final chapter of this thesis we want to take a look back, give an overview of the results we have achieved so far and set them into context. Even though we extended the Nash flow over time model by a fair amount, this is by no means a complete theory yet. Therefore, we want to also have a look into the future and give an outlook on further research as well as discuss a number of problems that remain open.

8.1 Review

Motivated by the dynamic traffic assignment problem and with the goal in mind to obtain a better understanding of the complex traffic dynamics, we considered a dynamic game in a flow over time model with deterministic queuing. The dynamic equilibria, called Nash flows over time, in the base version of this model was already understood quite well, but is only a very rough approximation of a real-world traffic scenario. In order to close the gap between large-scale simulation tools, which work well in practice but lack any provable foundation, and the mathematical theory, we extended this base model by several very natural traffic features and showed that, by extending the proof ideas, we could, in most cases, preserve the existence and the structural insights of dynamic equilibria.
As a first starting point we introduced time-dependent capacities and time-dependent speed limits in order to represent changes in the road network, such as planned construction work or school zones. This extension changes the base model only slightly and it is no surprise that it is possible to transfer the concept of thin flows with resetting and the construction of Nash flows over time with only minor adjustments.
Much more challenging was the consideration of multiple commodities in such a dynamic network game. The essential property of a global FIFO principle does not hold for these scenarios, and therefore, it is not possible anymore to extend multi-commodity Nash flows over time step by step. Instead, we have to consider all infinitesimally small players at the same time as the choice of early particles depends not only on all the flow already in the network but may also depend on all future flow. Even though we could not use any of the techniques of the base model, we were still able to show the existence of dynamic equilibria in this multi-commodity setting with the help of infinite dimensional-variational inequalities. Since this was known prior to the work in this thesis, the main contribution is the structural insight into these Nash flows over time, as we could show that their derivatives have to satisfy a set of conditions similar to the thin flow equations. The major difference to the single-commodity case is that we cannot consider each thin flow isolated anymore, but instead, we have to take into account the flow of the other commodities (the so-called foreign flow), and therefore, we have to consider all flow from the past and the future simultaneously. Unfortunately, this still does not give a clear instruction on how to construct Nash flows over time with multiple commodities, however, for the special case that all commodities share the same origin we showed that the problem of constructing a Nash flow over time reduces to the single-commodity case. The same holds true for the other extreme case, that every particle can start at multiple origins but they all share a common destination.

Clearly, the extension of the base model to a model with spillback and kinematic waves is one of the main contributions of this thesis. These fundamental traffic features, which are especially relevant in highly congested networks, were a huge challenge for the deterministic queuing model. The key idea is to restrict the total amount of flow on each arc by an arc specific storage capacity and to model the backwards moving gaps between vehicles by a gap flow over time. In order to obtain a well-defined flow over time based on the route choices of the particles we introduced a priority rule on each intersection. The fair allocation condition guarantees that, in the case of spillback, the particles merge according to the outflow capacity of each incoming link. As we only considered a single-commodity we could, again, use the network-wide FIFO principle in order to construct a Nash flow over time in this kinematic wave model. However, most of the proof ideas do not transfer to this extension, and therefore, most of the proofs became much more involved. In addition, we observed that the arrival times of these dynamic equilibria are not unique anymore and that, due to the fact that congestions can block intersections, the price of anarchy is unbounded in the spillback model.
Finally, we considered a different type of flows over time, called *instantaneous dynamic equilibria*. Here, particles do not predict the future evolution, but instead, each of them chooses a route depending on the current network configuration, i.e., the current shortest distance to the destination. As these might change drastically along the way, each player is allowed to adopt his or her route choice on the way. Even though these flows over time do not form a game theoretical Nash equilibrium, they are well motivated by real-world scenarios since traffic users, following a navigation system, might get re-routed as soon as the current traffic conditions change. As we only have to consider the current waiting times it becomes much easier to construct such IDE flows. In fact, we could even handle multi-commodity IDE flows with the same thin flow technique we used for single-commodity Nash flows over time.

8.2 Open Problems

Even though we have achieved quite a lot already, there still remain several open problems for further research. We want to mention the most prominent future challenges on Nash flows over time in the following.

Number of thin flow phases. We conjecture that, at least for the base model, the number of thin flow phases is finite for every network. It has been shown that this number can be exponential in the network size [19], but so far, it is not even clear whether there is a network where the extension step sizes converge to 0, or if there exists some network-dependent $\varepsilon > 0$ such that each thin flow phase has at least length ε. In all our experiments, however, the Nash flow over time had a final phase that lasts indefinitely.
This question is very relevant for computing Nash flows over time algorithmically. If there exists an instance that produces infinitely many phases, then the corresponding dynamic equilibrium can most likely not be described by a finite description, and hence, it cannot be computed algorithmically.

Computation complexity of thin flows. Related to the number of thin flow phases is the question of the complexity of computing a single thin flow with resetting. The mixed integer formulation for the base model, which we discussed in Section 3.6.1, shows that it is possible to compute a thin flow in exponential running time and that this problem lies in the complexity class NP. However, it remains a challenging open problem whether this problem is NP-hard, meaning that we cannot hope for a fast algorithm, or if it can be computed in polynomial time.

Price of anarchy. For the kinematic wave model it is very easy to see that all versions of the price of anarchy are unbounded, which can already be shown for very simple network topologies; see Figure 6.6 on page 114. For the base model, however, it remains an open problem for now. With the latest progress by Correa et al. in [20] it seems as if it is only a small step to show a bound of $\frac{e}{e-1}$ for the time price of anarchy, but the related monotonicity conjecture (Conjecture 3.12 on page 42) turns out to be more challenging to prove than expected.

Uniqueness of earliest arrival times. In the kinematic wave model we could show that spillback thin flows do not have unique ℓ'-labels, and hence, there exist networks with multiple Nash flows over time with different earliest arrival times. In the base model, however, the ℓ'-labels of a thin flow with resetting are uniquely determined, but it still remains an open question whether this translates to unique earliest arrival times. For more details on this topic we refer to Section 3.6.2 on page 41.

Structure of multi-commodity Nash flows over time. As a first step, we showed in Section 5.1.3 that the derivatives of a multi-commodity Nash flow over time are characterized by the multi-commodity thin flow conditions. Unfortunately, the properties of the x'- and ℓ'-functions are not obvious. We conjecture that the earliest arrival times ℓ must be piece-wise linear, or in other words, the ℓ' functions must be piece-wise constant. The intuition behind this is the following consideration. Whenever a current shortest path changes for some commodity, this means that either an arc became active or a queue depleted. It seems that every arc can only be responsible for a countable amount of events and every event will cause at most a countable amount of jump points for the ℓ'-functions. Unfortunately, we were not able to prove this yet.

Long-term behavior of Nash flows over time in the kinematic wave model. For the base model the long-term behavior of Nash flows over time is well understood as long as the minimal s-t-cut is smaller than the network inflow rate. In fact, a dynamic equilibrium reaches a steady state if, and only if, this cut condition is satisfied. A characterization of the long-term behavior of a Nash flow over time in the spillback or kinematic wave model remains an open problem. Is it even possible to denote simple conditions on the network in order to guarantee a steady state? Due to the nature of spillback, Nash flows over time are even harder to predict in this extended model.

IDE flows in the spillback model. A probably simpler open problem is the question whether the concept of an IDE flow can be transferred to the spillback, or even to the kinematic wave, model. As the effective capacity $c_v \cdot \nu_e^-$ of an arc e can change drastically in the spillback model, it is not obvious what information is available for particles to choose their routes. In the model without spillback we use the current waiting times of each arc, which are the actual waiting times for particles entering the link at this very moment. In a spillback model, however, the actual waiting times heavily depend on the future evolution of the flow over time since arcs further downstream might become full and restrict the outflow rates. Hence, the first step to transfer IDE flows to the spillback model would be to consider a reasonable information model.

Combination of IDE flows and Nash flows over time. The main difference between IDE flows and Nash flows over time is that the particles of an IDE flow only consider the current waiting times, whereas the particles of a Nash flow over time predict the complete future flow evolution. Hence, an obvious question is, whether there exists a combined flow over time model with a parameter $H \in [0, \infty)$, which describes how far particles predict the future. For a well-defined model these equilibrium flows should be equal to IDE flows for $H = 0$ and equal to Nash flows over time for an H that is larger than the largest travel time from the source to the sink. The open questions, what properties such equilibrium flows would have or whether they even exist, have not been studied so far.

Convergence of packet routing models to non-atomic flows over time. In the real world, traffic does not consist of a continuous flow that can be split up arbitrarily, but instead, we have a set of atomic, non-splittable vehicles. This is captured in so-called packet routing models, which have been considered a lot. Surprisingly, the relation between these atomic models and non-atomic flows over time has not been studied very intensively so far. As a first step, it would be interesting to show that there exists a packet routing model with discrete time steps that converges to a flow over time in the deterministic queuing model, when the packet size and the time step size go to zero. It would be very interesting to study dynamic equilibria in such a packet routing model and maybe it could even be possible to show that these equilibria also converge to a Nash flow over time.

Regulating dynamic equilibria by tolls. Tolls are an obvious tool for an authority to regulate traffic. It would be very interesting to consider dynamic equilibria in a flow over time model with tolls. Unfortunately, it seems that for fixed tolls a dynamic equilibrium would not implement a global FIFO principle anymore since particles on roads with tolls might overtake particles on toll-free routes. By some first considerations it seems that the existence of single-commodity Nash flows over time with tolls is as complicated as it is to prove the existence of multi-commodity Nash flows over time without tolls. Nonetheless, this is a very relevant research direction. For example, it might be possible to implement tolls that force a dynamic equilibrium to be optimal, i.e., to be an earliest arrival flow. For some first results in this regard we refer to the very recent preprint article of Frascaria and Olver [34].

Further open questions. These are only some of the obvious open questions and there are many more. For example, questions about a characterization of Braess arcs, about the minimal investment into storage capacities in order to avoid spillback or about the robustness of Nash flows over time if not all particles act rationally, just to name a few. Flow over time models with deterministic queuing, and especially Nash flows over time, remain an interesting and active field of research with, hopefully, a lot more to come.

8.3 Conclusion

It is fair to say that the contribution of the work presented in this thesis is a first but important step in order to obtain a better mathematical understanding of the dynamic traffic assignment problem. We provide important structural insights into dynamic equilibria in order to build a solid mathematical foundation, which is not only interesting from an academical perspective but also useful to improve and evaluate large-scale simulation tools used for real-world scenarios. In order to improve the traffic infrastructure and to reduce congestions in highly populated regions it is essential that network designers can rely on accurate forecasts, which can be provided by such simulations.

Even though this is only a small part in the overall picture and there are a lot of open problems remaining, the research work will continue, and hopefully sooner than later, mathematics, and science as a whole, will have a positive impact on one of urgent problems of today's society, the climate change. It might be too late to prevent most of the drastic consequences, but nonetheless, science needs to try its best to reduce anthropogenic emissions of atmospheric green house gases in order to temper the impact on our society and on nature as a whole. **There is no planet B.**

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